1. Evaluate $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ for each of the following vector fields.

(a) $\mathbf{F} = x\hat{i} + y\hat{j} + z\hat{k}$
(b) $\mathbf{F} = xy^2\hat{i} - yz^2\hat{j} + zx^2\hat{k}$
(c) $\mathbf{F} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2 + y^2}}$ (the polar basis vector $\hat{r}$ in 2d)
(d) $\mathbf{F} = \frac{-yk^2}{\sqrt{x^2 + y^2}^3}$ (the polar basis vector $\hat{\theta}$ in 2d)

**Solution.** (a) By definition

$$\nabla \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$\nabla \times (x\hat{i} + y\hat{j} + z\hat{k}) = \text{det} \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{bmatrix} = 0$$

(b) By definition

$$\nabla \cdot (xy^2\hat{i} - yz^2\hat{j} + zx^2\hat{k}) = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(-yz^2) + \frac{\partial}{\partial z}(zx^2) = y^2 - z^2 + x^2$$

$$\nabla \times (xy^2\hat{i} - yz^2\hat{j} + zx^2\hat{k}) = \text{det} \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & -yz^2 & zx^2 \end{bmatrix} = 2yz\hat{i} - 2xz\hat{j} - 2xy\hat{k}$$

(c) By definition

$$\nabla \cdot \left(\frac{x}{\sqrt{x^2 + y^2}}\hat{i} + \frac{y}{\sqrt{x^2 + y^2}}\hat{j}\right) = \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2}}\right) + \frac{\partial}{\partial y} \left(\frac{y}{\sqrt{x^2 + y^2}}\right) = 1$$

$$\nabla \times \left(\frac{x}{\sqrt{x^2 + y^2}}\hat{i} + \frac{y}{\sqrt{x^2 + y^2}}\hat{j}\right) = \text{det} \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} & 0 \end{bmatrix} = -\frac{xy}{[x^2 + y^2]^{3/2}} + \frac{xy}{[x^2 + y^2]^{3/2}}\hat{k} = 0$$

(d) By definition

$$\nabla \cdot \left(-\frac{y}{\sqrt{x^2 + y^2}}\hat{i} + \frac{x}{\sqrt{x^2 + y^2}}\hat{j}\right) = \frac{\partial}{\partial x} \left(-\frac{y}{\sqrt{x^2 + y^2}}\right) + \frac{\partial}{\partial y} \left(\frac{x}{\sqrt{x^2 + y^2}}\right)$$

$$\nabla \times \left(-\frac{y}{\sqrt{x^2 + y^2}}\hat{i} + \frac{x}{\sqrt{x^2 + y^2}}\hat{j}\right) = \text{det} \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{\sqrt{x^2 + y^2}} & \frac{x}{\sqrt{x^2 + y^2}} & 0 \end{bmatrix} = \left(\frac{1}{\sqrt{x^2 + y^2}} \frac{x^2}{[x^2 + y^2]^{3/2}} + \frac{1}{\sqrt{x^2 + y^2}} \frac{y^2}{[x^2 + y^2]^{3/2}}\right)\hat{k} = \frac{1}{\sqrt{x^2 + y^2}}$$
2. Does \( \nabla \times \mathbf{F} \) have to be perpendicular to \( \mathbf{F} \)?

**Solution.** [No]. The field in part (b) of the last question provides a counterexample.

3. Verify the vector identities
   
   (a) \( \nabla \cdot (f \mathbf{F}) = f \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f \)
   
   (b) \( \nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \)
   
   (c) \( \nabla^2 (fg) = f \nabla^2 g + 2 \nabla f \cdot \nabla g + g \nabla^2 f \)

**Solution.** (a) By the product rule

\[
\nabla \cdot (f \mathbf{F}) = \frac{\partial}{\partial x} (fF_1) + \frac{\partial}{\partial y} (fF_2) + \frac{\partial}{\partial z} (fF_3)
\]

\[
= f \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} + F_1 \frac{\partial f}{\partial x} + F_2 \frac{\partial f}{\partial y} + F_3 \frac{\partial f}{\partial z}
\]

\[
= f \nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f
\]

(b) Again by the product rule

\[
\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \frac{\partial}{\partial x} (F_2G_3 - F_3G_2) + \frac{\partial}{\partial y} (F_3G_1 - F_1G_3) + \frac{\partial}{\partial z} (F_1G_2 - F_2G_1)
\]

\[
= \frac{\partial F_2}{\partial x} G_3 - \frac{\partial F_3}{\partial x} G_2 + \frac{\partial F_3}{\partial y} G_1 - \frac{\partial F_1}{\partial y} G_3 + \frac{\partial F_1}{\partial z} G_2 - \frac{\partial F_2}{\partial z} G_1
\]

\[
+ F_1 \frac{\partial G_3}{\partial y} - F_3 \frac{\partial G_2}{\partial y} + F_3 \frac{\partial G_1}{\partial y} - F_1 \frac{\partial G_3}{\partial y} + F_1 \frac{\partial G_2}{\partial z} - F_2 \frac{\partial G_1}{\partial z}
\]

\[
= (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}) G_1 + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}) G_2 + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) G_3
\]

\[
= \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})
\]

(c) Recall that \( \nabla^2 (fg) = \nabla \cdot [\nabla (fg)] \). First

\[
\nabla (fg) = i \frac{\partial}{\partial x} (fg) + j \frac{\partial}{\partial y} (fg) + k \frac{\partial}{\partial z} (fg)
\]

\[
= ig \frac{\partial f}{\partial x} + jf \frac{\partial g}{\partial y} + kg \frac{\partial f}{\partial z}
\]

\[
+ if \frac{\partial g}{\partial x} + jg \frac{\partial f}{\partial y} + kf \frac{\partial g}{\partial z}
\]

\[
= g \nabla f + f \nabla g
\]

So by part (a), twice,

\[
\nabla^2 (fg) = \nabla \cdot (g \nabla f) + \nabla \cdot (f \nabla g)
\]

\[
= g (\nabla \cdot \nabla f) + (\nabla g) \cdot (\nabla f) + f (\nabla \cdot \nabla g) + (\nabla f) \cdot (\nabla g)
\]

\[
= f \nabla^2 g + 2 \nabla f \cdot \nabla g + g \nabla^2 f
\]

4. A rigid body rotates at an angular velocity of \( \Omega \) rad/sec about an axis that passes through the origin and has direction \( \hat{a} \). When you are standing at the head of \( \hat{a} \) looking towards the origin, the rotation is counterclockwise. Set \( \omega = \Omega \hat{a} \).
5. **(Optional — not to be handed in)** Find the speed of the students in a classroom located at latitude 49° N due to the rotation of the Earth. Ignore the motion of the Earth about the Sun, the Sun in the Galaxy and so on. The radius of the Earth is 6378 km.

**Solution.** The students are a distance 6378 sin(49°) = 4184 km from the axis of rotation. The rate of rotation is \( \Omega = \frac{2 \pi}{24} \) radians per hour. In each hour the students sweep out an arc of \( \frac{2 \pi}{24} \) radians from a circle of radius 4184 km. Their speed is \( \frac{2 \pi}{24} \times 4184 = 1095 \text{ km/hr} \).

6. Find, if possible, a vector field \( A \) that has \( \mathbf{k} \) component \( A_3 = 0 \) and that is a vector potential for

(a) \( \mathbf{F} = (1 + yz) \mathbf{i} + (2y + zx) \mathbf{j} + (3z^2 + xy) \mathbf{k} \)

(b) \( \mathbf{G} = yz \mathbf{i} + zx \mathbf{j} + xy \mathbf{k} \)

**Solution.** (a) Since \( \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (1 + yz) + \frac{\partial}{\partial y} (2y + zx) + \frac{\partial}{\partial z} (3z^2 + xy) = 2 + 6z \neq 0, \) \( \mathbf{F} \) fails the screening test and cannot have a vector potential.

(b) The vector field \( \mathbf{A} = A_1 \mathbf{i} + A_2 \mathbf{j} \) is a vector potential for \( \mathbf{G} \) if and only if \( \mathbf{G} = \nabla \times \mathbf{A} \), which is the case if and only if

\[
\begin{align*}
-\frac{\partial A_x}{\partial z} &= yz & \quad & \iff & \quad A_x &= \frac{1}{2} yz^2 + B_2(x, y) \\
\frac{\partial A_z}{\partial x} &= zx & \quad & \iff & \quad A_z &= \frac{1}{2} xz^2 + B_1(x, y) \\
\frac{\partial A_y}{\partial x} &= xy & \quad & \iff & \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= xy
\end{align*}
\]
There are infinitely many solutions to \( \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} = xy \). In fact \( B_2 \) is completely arbitrary. If one chooses \( B_2 = 0 \), then \( B_1 = -\frac{1}{4}xy^2 \) does the job. If one chooses \( B_1 = 0 \), then \( B_2 = \frac{1}{4}x^2y \) does the job. Thus two solutions are \( A = \frac{1}{4}(x^2 - y^2)\hat{i} - \frac{1}{2}yz\hat{j} \) and \( A = \frac{1}{4}x^2\hat{i} + \frac{1}{2}(x^2 - y^2)y\hat{j} \).

7. \( \text{(Optional — not to be handed in)} \) Suppose that the vector field \( \mathbf{F} \) obeys \( \nabla \cdot \mathbf{F} = 0 \) in all of \( \mathbb{R}^3 \).

Let \( \mathbf{r}(t) = t\hat{i} + ty\hat{j} + tz\hat{k}, 0 \leq t \leq 1 \) be a parametrization of the line segment from the origin to \((x, y, z)\).

Define

\[
\mathbf{G}(x, y, z) = \int_0^1 t\mathbf{F}(\mathbf{r}(t)) \times \frac{d\mathbf{r}}{dt}(t) \, dt
\]

Show that \( \nabla \times \mathbf{G} = \mathbf{F} \) throughout \( \mathbb{R}^3 \).

**Solution.** We shall show that \( \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = F_1 \). The other components are similar. First we have

\[
t\mathbf{F}(\mathbf{r}(t)) \times \frac{d\mathbf{r}}{dt}(t) = t\mathbf{F}(tx, ty, tz) \times (x\hat{i} + y\hat{j} + z\hat{k}) = t \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ F_1 & F_2 & F_3 \\ x & y & z \end{bmatrix}
\]

Reading off the \( \hat{k} \) and \( \hat{j} \) components of the determinant gives

\[
G_3(x, y, z) = \int_0^1 t[F_1(tx, ty, tz)y - F_2(tx, ty, tz)x] \, dt
\]

\[
G_2(x, y, z) = \int_0^1 t[F_3(tx, ty, tz)x - F_1(tx, ty, tz)z] \, dt
\]

So

\[
\frac{\partial G_3}{\partial y} = \int_0^1 t [F_1(tx, ty, tz) + \frac{\partial F_1}{\partial y}(tx, ty, tz)ty - \frac{\partial F_2}{\partial y}(tx, ty, tz)tx] \, dt
\]

\[
\frac{\partial G_2}{\partial z} = \int_0^1 t [\frac{\partial F_3}{\partial z}(tx, ty, tz)tx - \frac{\partial F_1}{\partial z}(tx, ty, tz)tz - F_1(tx, ty, tz)] \, dt
\]

\[
\Rightarrow \frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = \int_0^1 \left[ 2tF_1(tx, ty, tz) + t^2y\frac{\partial F_1}{\partial y}(tx, ty, tz) + t^2z\frac{\partial F_1}{\partial z}(tx, ty, tz)
\right.
\]

\[
\left. - t^2x\frac{\partial F_2}{\partial y}(tx, ty, tz) - t^2x\frac{\partial F_3}{\partial z}(tx, ty, tz) \right] \, dt
\]

Since, by hypothesis, \( \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 \), the last two terms

\[
-t^2x\left\{ \frac{\partial F_2}{\partial y}(tx, ty, tz) + \frac{\partial F_3}{\partial z}(tx, ty, tz) \right\} = -t^2x\left\{ - \frac{\partial F_1}{\partial x}(tx, ty, tz) \right\}
\]

so that

\[
\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = \int_0^1 \left[ 2tF_1(tx, ty, tz) + t^2x\frac{\partial F_1}{\partial x}(tx, ty, tz) + t^2y\frac{\partial F_1}{\partial y}(tx, ty, tz) + t^2z\frac{\partial F_1}{\partial z}(tx, ty, tz) \right] \, dt
\]

\[
= \int_0^1 \frac{d}{dt}\left[ t^2F_1(tx, ty, tz) \right] \, dt = \left[ t^2F_1(tx, ty, tz) \right]_{t=0}^{t=1} = F_1(x, y, z)
\]