1. Find the velocity, speed and acceleration at time \( t \) of the particle whose position is
\[
r(t) = a \cos t \hat{i} + a \sin t \hat{j} + ct \hat{k}
\]

Describe the path of the particle.

Solution.
\[
r(t) = a \cos t \hat{i} + a \sin t \hat{j} + ct \hat{k}
\]
\[
v(t) = r'(t) = -a \sin t \hat{i} + a \cos t \hat{j} + c \hat{k}
\]
\[
\frac{ds}{dt}(t) = |r'(t)| = \sqrt{a^2 + c^2}
\]
\[
a(t) = r''(t) = -a \cos t \hat{i} - a \sin t \hat{j}
\]

As \( t \) runs over an interval of length \( 2\pi \), \((x, y)\) traces out a circle of radius \( a \) and \( z \) increases by \( 2\pi c \). The path is a helix with radius \( a \) and with each turn having height \( 2\pi c \).

2. A projectile falling under the influence of gravity and slowed by air resistance proportional to its speed has position satisfying
\[
\frac{d^2 r}{dt^2} = -g \hat{k} - \alpha \frac{dr}{dt}
\]
where \( \alpha \) is a positive constant. If \( r = r_0 \) and \( \frac{dr}{dt} = v_0 \) at time \( t = 0 \), find \( r(t) \). (Hint: Define \( u(t) = e^{\alpha t} \frac{dr}{dt}(t) \) and substitute \( \frac{dr}{dt}(t) = e^{-\alpha t} u(t) \) into the given differential equation to find a differential equation for \( u \).)

Solution. Define \( u(t) = e^{\alpha t} \frac{dr}{dt}(t) \). Then
\[
\frac{du}{dt}(t) = \alpha e^{\alpha t} \frac{dr}{dt}(t) + e^{\alpha t} \frac{d^2 r}{dt^2}(t)
\]
\[
= \alpha e^{\alpha t} \frac{dr}{dt}(t) - ge^{\alpha t} \hat{k} - \alpha e^{\alpha t} \frac{dr}{dt}(t)
\]
\[
= -ge^{\alpha t} \hat{k}
\]

Integrating both sides of this equation from \( t = 0 \) to \( t = T \) gives
\[
u(T) - u(0) = -g \frac{e^{\alpha T} - 1}{\alpha} \hat{k} \Rightarrow u(T) = u(0) - g \frac{e^{\alpha T} - 1}{\alpha} \hat{k} = \frac{dr}{dt}(0) - g \frac{e^{\alpha T} - 1}{\alpha} \hat{k} = v_0 - g \frac{e^{\alpha T} - 1}{\alpha} \hat{k}
\]

Subbing in \( u(T) = e^{\alpha t} \frac{dr}{dt}(T) \) and multiplying through by \( e^{-\alpha T} \)
\[
\frac{dr}{dt}(T) = e^{-\alpha T} v_0 - g \frac{1 - e^{-\alpha T}}{\alpha} \hat{k}
\]

Integrating both sides of this equation from \( T = 0 \) to \( T = t \) gives
\[
r(t) - r(0) = \frac{e^{-\alpha t} - 1}{-\alpha} v_0 - g \frac{t}{\alpha} \hat{k} + g \frac{e^{-\alpha t} - 1}{-\alpha^2} \hat{k}
\]

\[
\Rightarrow r(t) = r_0 + \frac{e^{-\alpha t} - 1}{\alpha} v_0 + g \frac{1 - \alpha t - e^{-\alpha t}}{\alpha^2} \hat{k}
\]
3. Find the specified parametrization of the first quadrant part of the circle \( x^2 + y^2 = a^2 \).
   
   (a) In terms of the \( y \) coordinate.
   
   (b) In terms of the angle between the tangent line and the positive \( x \)-axis.
   
   (c) In terms of the arc length from \((0, a)\).

**Solution.**

(a) Since \( y = \sqrt{a^2 - x^2} \), the parametrization is
\[
(x(t), y(t)) = \left( \sqrt{a^2 - t^2}, t \right), \ 0 \leq t \leq a
\]

(b) Let \( \theta \) be the angle between the radius vector \((a \cos \theta, a \sin \theta)\) and the positive \( x \)-axis. The tangent line to the circle at \((a \cos \theta, a \sin \theta)\) is perpendicular to the radius vector and so makes angle \( \phi = \frac{\pi}{2} + \theta \) with the positive \( x \) axis. (See the figure on the left below.) As \( \theta = \phi - \frac{\pi}{2} \), the desired parametrization is
\[
(x(\phi), y(\phi)) = (a \cos(\phi - \frac{\pi}{2}), a \sin(\phi - \frac{\pi}{2})) = (a \sin \phi, -a \cos \phi), \ \frac{\pi}{2} \leq \phi \leq \pi
\]

(c) Let \( \theta \) be the angle between the radius vector \((a \cos \theta, a \sin \theta)\) and the positive \( x \)-axis. The arc from \((0, a)\) to \((a \cos \theta, a \sin \theta)\) subtends an angle \( \frac{\pi}{2} - \theta \) and so has length \( s = a(\frac{\pi}{2} - \theta) \). (See the figure on the right above.) Thus \( \theta = \frac{\pi}{2} - \frac{s}{a} \) and the desired parametrization is
\[
(x(s), y(s)) = (a \cos(\frac{\pi}{2} - \frac{s}{a}), a \sin(\frac{\pi}{2} - \frac{s}{a})), \ 0 \leq s \leq \frac{\pi}{2} a
\]

4. Find the length of the parametric curve
\[
x = a \cos t \sin t \quad y = a \sin^2 t \quad z = bt
\]

between \( t = 0 \) and \( t = T > 0 \).

**Solution.**

\[
x'(t) = a \left[ \cos^2 t - \sin^2 t \right] = a \cos 2t
\]
\[
y'(t) = 2a \sin t \cos t = a \sin 2t
\]
\[
z'(t) = b
\]

So
\[
\frac{dx}{dt}(t) = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{a^2 + b^2} \quad \Rightarrow \quad \text{length} = \sqrt{a^2 + b^2} \cdot T
\]
5. Reparametrize the curve

\[ \mathbf{r}(t) = a \cos^3 t \mathbf{i} + a \sin^3 t \mathbf{j} + b \cos 2t \mathbf{k}, \quad 0 \leq t \leq \frac{\pi}{2} \]

with the same orientation, in terms of arc length measured from the point where \( t = 0 \). You may use the formulae

\[ \sin(2t) = 2 \sin t \cos t \quad 1 - \cos(2t) = 2 \sin^2(t) \]

to simplify the computations.

**Solution.**

\[ \mathbf{v}(t) = \mathbf{r}'(t) = (-3a \cos^2 t \sin t, 3a \sin^2 t \cos t, -2b \sin 2t) \]
\[ \Rightarrow \quad \frac{ds}{dt} = |\mathbf{v}(t)| = \left[ 9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t + 4b^2 \sin^2 2t \right]^{1/2} \]
\[ = \left[ 9a^2 \sin^2 t \cos^2 t + 4b^2 \sin^2 2t \right]^{1/2} \]
\[ = \left[ \frac{9}{4}a^2 + 4b^2 \right]^{1/2} |\sin 2t| = \left[ \frac{9}{4}a^2 + 4b^2 \right]^{1/2} \sin 2t \text{ for } 0 \leq t \leq \frac{\pi}{2} \]

Integrating (and recalling that \( s = 0 \) corresponds to \( t = 0 \))

\[ s(t) = \left[ \frac{9}{4}a^2 + 4b^2 \right]^{1/2} \frac{1 - \cos 2t}{2} = \left[ \frac{9}{4}a^2 + 4b^2 \right]^{1/2} \sin^2 t \]

Setting \( K = \left[ \frac{9}{4}a^2 + 4b^2 \right]^{1/2} \), we have \( \sin t = \frac{\sqrt{K}}{\sqrt{K}} \), \( \cos t = \sqrt{1 - \frac{K}{K}} \), \( \cos 2t = 1 - 2\sin^2 t = 1 - \frac{2K}{K} \) and hence

\[ \mathbf{r}(s) = a \left[ 1 - \frac{K}{\sqrt{K}} \right]^{3/2} \mathbf{i} + a \left[ \frac{K}{\sqrt{K}} \right]^{3/2} \mathbf{j} + b \left[ 1 - \frac{2K}{\sqrt{K}} \right] \mathbf{k}, \quad 0 \leq s \leq K, \text{ where } K = \left[ \frac{9}{4}a^2 + 4b^2 \right]^{1/2} \]

6. The plane \( z = 2x + 3y \) intersects the cylinder \( x^2 + y^2 = 9 \) in an ellipse. Find a parametrization of the ellipse. Express the circumference of this ellipse as an integral. You need not evaluate the integral.

**Solution.** We can parametrize the circle \( x^2 + y^2 = 9 \) as \( x(\theta) = 3 \cos \theta, \quad y(\theta) = 3 \sin \theta \) with \( \theta \) running from 0 to \( 2\pi \). As \( z = 2x + 3y \), the ellipse can be parametrized by

\[ x(\theta) = 3 \cos \theta, \quad y(\theta) = 3 \sin \theta, \quad z(\theta) = 6 \cos \theta + 9 \sin \theta, \quad 0 \leq \theta \leq 2\pi \]

As

\[ \frac{ds}{d\theta} = \sqrt{x'(\theta)^2 + y'(\theta)^2 + z'(\theta)^2} \]
\[ = \sqrt{9 \sin^2 \theta + 9 \cos^2 \theta + 36 \sin^2 \theta + 81 \cos^2 \theta - 108 \sin \theta \cos \theta} \]
\[ = \sqrt{45 + 45 \cos^2 \theta - 108 \sin \theta \cos \theta} \]

The circumference is

\[ s = \int_0^{2\pi} \sqrt{45 + 45 \cos^2 \theta - 108 \sin \theta \cos \theta} \, d\theta \]

7. A wire of total length 1000 cm is formed into a flexible coil that is a circular helix. If there are 10 turns to each centimeter of height and the radius of the helix is 3 cm, how tall is the coil?

**Solution.** The parametrized equation of a helix is

\[ \mathbf{r}(\theta) = a \cos \theta \mathbf{i} + a \sin \theta \mathbf{j} + b \theta \mathbf{k} \]
The radius of the helix is 3 cm, so $a = 3$ cm. After 10 turns (i.e. $\theta = 20\pi$) the height is 1 cm, so $b = \frac{1}{20\pi}$ cm/rad. Thus $\mathbf{r}(\theta) = 3\cos \theta \mathbf{i} + 3\sin \theta \mathbf{j} + \frac{1}{20\pi} \theta \mathbf{k}$ and $\mathbf{r}'(\theta) = -3\sin \theta \mathbf{i} + 3\cos \theta \mathbf{j} + \frac{1}{20\pi} \mathbf{k}$ so that $rac{ds}{d\theta} = |\mathbf{r}'(\theta)| = \sqrt{9 + \frac{1}{400\pi^2}}$. If $\theta$ varies from $\theta = 0$ to $\theta = \theta_F$, then the wire has length $\sqrt{9 + \frac{1}{400\pi^2}} \theta_F$. This must be 1000 cm. So $\theta_F = 1000\left[9 + \frac{1}{400\pi^2}\right]^{-1/2}$. This corresponds to a height $b \theta_F = \frac{1}{20\pi}1000\left[9 + \frac{1}{400\pi^2}\right]^{-1/2} \approx 5.3$ cm.

8. You are lost in a desert during the night. There is a road as indicated in the figure on the next page. Your position is (100, 190). A car is approaching from the left with headlights that have range 70m. Will the driver be able to see you?

**Solution.** In order for the driver to see you when the car is at the point $(x_0, y_0)$ on the road, two conditions have to be satisfied.

- $(x_0, y_0)$ has to be within a distance of 70m from (100, 190). That is, $(x_0, y_0)$ must be inside the circle of radius 70m centred on (100, 190). That is the circle in the figure on the next page.
- The headlights have to be pointing in the correct direction. The headlights are pointing in the direction of the tangent to the curve at $(x_0, y_0)$, so the tangent line to the curve at $(x_0, y_0)$ must pass through (100, 190).

Now look at the figure on the next page. The second condition forces $(x_0, y_0)$ to be near $(0, 0)$. Because the distance from $(0, 0)$ to (100, 190) is much greater than 70m, the driver will not be able to see you.