Theorem Let the vector field $\vec{F}$ be defined and have continuous first partial derivatives on all of $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\vec{F}$ is conservative if and only if it passes the screening test

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \quad \text{in } \mathbb{R}^2$$

$$\vec{\nabla} \times \vec{F} = \vec{0} \quad \text{in } \mathbb{R}^3$$

Proof: Just do 2d.

Already know conservative $\Longrightarrow$ passes screening test

So assume $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$

Need to find a $\phi(x,y)$ obeying

① $\frac{\partial \phi}{\partial x}(x,y) = F_1(x,y)$

② $\frac{\partial \phi}{\partial y}(x,y) = F_2(x,y)$

① $\Rightarrow$ $\phi(x,y) = \int_0^x F_1(X,y)\, dX + f(y)$

② $\Rightarrow$ $\frac{\partial}{\partial y}\left[\int_0^x F_1(X,y)\,dX + f(y)\right] = \int_0^x \frac{\partial F_1}{\partial y}(X,y)\,dX + f'(y) = F_2(x,y)$

$$\Rightarrow \quad f'(y) = F_2(x,y) - \int_0^x \frac{\partial F_1}{\partial y}(X,y)\,dX$$

screening test

$$= F_2(x,y) - \int_0^x \frac{\partial F_2}{\partial X}(X,y)\,dX$$

FToC

$$= F_2(x,y) - F_2(X,y)\Big|_0^x$$

$$= F_2(x,y) - F_2(x,y) + F_2(0,y) = F_2(0,y)$$

$$\Rightarrow \quad f(y) = \int_0^y F_2(0,Y)\,dY + C \quad \text{works}$$

$\blacksquare$