MATH 317 PROBLEM SET VII

Due March 19

1) Let $\vec{F} = (x - yz)\hat{\imath} + (y + xz)\hat{\jmath} + (z + 2xy)\hat{k}$ and let
   S_1 be the portion of the cylinder $x^2 + y^2 = 2$ that lies inside the sphere $x^2 + y^2 + z^2 = 4$
   S_2 be the portion of $x^2 + y^2 + z^2 = 4$ that lies outside the cylinder $x^2 + y^2 = 2$
   V be the volume bounded by S_1 and S_2

   Compute
   a) $\iint_{S_1} \vec{F} \cdot \hat{n}\, dS$ with $\hat{n}$ pointing inward
   b) $\iiint_V \vec{\nabla} \cdot F\, dV$
   c) $\iint_{S_2} \vec{F} \cdot \hat{n}\, dS$ with $\hat{n}$ pointing outward

   Use the divergence theorem to answer at least one of parts (a), (b) and (c).

2) Evaluate the integral $\iint_S \vec{F} \cdot \hat{n}\, dS$, where $\vec{F} = (x, y, 1)$ and S is the surface $z = 1 - x^2 - y^2$, for $x^2 + y^2 \le 1$, by two methods.
   a) First, by direct computation of the surface integral.
   b) Second, by using the divergence theorem.

3a) By applying the divergence theorem to $\vec{F} = \phi\vec{a}$, where $\vec{a}$ is an arbitrary constant vector, show that

$$\iiint_V \vec{\nabla}\phi\, dV = \iint_{\partial V} \phi\hat{\mathbf{n}}\, dS$$

b) Show that the centroid $(\bar{x}, \bar{y}, \bar{z})$ of a solid V with volume $|V|$ is given by

$$(\bar{x}, \bar{y}, \bar{z}) = \tfrac{1}{2|V|} \iint_{\partial V} (x^2 + y^2 + z^2)\, \hat{\mathbf{n}}\, dS$$

4) Let V be the solid in 3-space defined by

$$0 \le z \le \frac{9 - x^2 - y^2}{9 + x^2 + y^2}$$

   Let S be the curved portion of the boundary of V oriented with outward normal and let

$$\vec{F} = zy^3\,\hat{\imath} + yx\,\hat{\jmath} + (2z + y^2)\hat{k}$$

   Assume that the volume of V is α and compute $\iint_S \vec{F} \cdot \hat{n}\, dS$ in terms of α.

5) Find the flux of $\vec{F} = (y + xz)\hat{\boldsymbol{\imath}} + (y + yz)\hat{\boldsymbol{\jmath}} - (2x + z^2)\hat{\mathbf{k}}$ upward through the first octant part of the sphere $x^2 + y^2 + z^2 = a^2$.

6) Let $\vec{E}(\vec{r})$ be the electric field due to a charge configuration that has density $\rho(\vec{r})$. Gauss' law states that, if V is any solid in $\mathbb{R}^3$ with surface ∂V, then the electric flux

$$\iint_{\partial V} \vec{E} \cdot \hat{\mathbf{n}}\, dS = 4\pi Q \qquad \text{where} \qquad Q = \iiint_V \rho\, dV$$

   is the total charge in V. Here, as usual, $\hat{\mathbf{n}}$ is the outward pointing unit normal to ∂V. Show that

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = 4\pi\rho(\vec{r})$$

   for all $\vec{r}$ in $\mathbb{R}^3$. This is one of Maxwell's equations. Assume that $\vec{\nabla} \cdot \vec{E}(\vec{r})$ and $\rho(\vec{r})$ are well–defined and continuous everywhere.

Reminder: Midterm II is scheduled for Friday, March 14.
Reminder: The final exam is on Tuesday, April 15 at 3:30pm.