1. Find the curvature of the plane curve \( y = e^x \).

2. Find the minimum and maximum values for the curvature of the ellipse

\[
x(t) = a \cos t, \quad y(t) = b \sin t
\]

Here \( a > b > 0 \).

3. Recall that a point with polar coordinates \( r \) and \( \theta \) has \( x = r \cos \theta \) and \( y = r \sin \theta \). Let \( r = f(\theta) \) be the equation of a plane curve in polar coordinates. Find the curvature of this curve at a general point \( \theta \).

4. Find the curvature of the cardioid \( r = a(1 - \cos \theta) \).

5. Find the unit tangent, unit normal and binormal vectors and the curvature and torsion of the curve

\[
r(t) = t \hat{i} + \frac{t^2}{2} \hat{j} + \frac{t^3}{3} \hat{k}
\]

6. Consider a curve that is parametrized by arc length \( s \).
   (a) Show that if the curve has curvature \( \kappa(s) = 0 \) for all \( s \), then the curve is a straight line.
   (b) Show that if the curve has curvature \( \kappa(s) > 0 \) and torsion \( \tau(s) = 0 \) for all \( s \), then the curve lies in a plane.
   (c) Show that if the curve has curvature \( \kappa(s) = \kappa_0 \), a strictly positive constant, and torsion \( \tau(s) = 0 \) for all \( s \), then the curve is a circle.

7. Suppose that the curve \( C \) is the intersection of the cylinder \( x^2 + y^2 = 1 \) with the surface \( z = x^2 - y^2 \).
   (a) Find a parameterization of \( C \).
   (b) Determine the curvature of \( C \) at the point \( (1/\sqrt{2}, 1/\sqrt{2}, 0) \).
   (c) Find the osculating plane to \( C \) at the point \( (1/\sqrt{2}, 1/\sqrt{2}, 0) \). In general, the osculating plane to a curve \( r(t) \) at the point \( r(t_0) \) is the plane which fits the curve best at \( r(t_0) \). It passes through \( r(t_0) \) and has normal vector \( \mathbf{B}(t_0) \).
   (d) Find the radius and the centre of the osculating circle to \( C \) at the point \( (1/\sqrt{2}, 1/\sqrt{2}, 0) \).

8. In this exercise, we make more precise the sense in which the osculating circle is the circle which best approximates a plane curve at a point.
   ○ By translating and rotating our coordinate system, we can always arrange that the point is \( (0,0) \) and that the curve is \( y = f(x) \) with \( f'(0) = 0 \) and \( f''(0) > 0 \). (We are assuming that the curvature at the point is nonzero.)
   ○ Let \( y = g(x) \) be the bottom half of the circle of radius \( r \) which is centred at \( (0, r) \).
   Show that if \( f(x) \) and \( g(x) \) have the same second order Taylor approximation at \( x = 0 \), then \( r \) is the radius of curvature of \( y = f(x) \) at \( x = 0 \).
9. (Optional — not to be handed in) A frictionless roller–coaster track has the form of one turn of the circular helix with parametrization \((a \cos \theta, a \sin \theta, b \theta)\). A car leaves the point where \(\theta = 2\pi\) with zero velocity and moves under gravity to the point where \(\theta = 0\). By Newton’s law of motion, the position \(\mathbf{r}(t)\) of the car at time \(t\) obeys

\[ m\mathbf{r}''(t) = \mathbf{N}(\mathbf{r}(t)) - mg\mathbf{k} \]

Here \(m\) is the mass of the car, \(g\) is a constant, \(-mg\mathbf{k}\) is the force due to gravity and \(\mathbf{N}(\mathbf{r}(t))\) is the force that the roller–coaster track applies to the car to keep the car on the track. Since the track is frictionless, \(\mathbf{N}(\mathbf{r}(t))\) is always perpendicular to \(\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}(t)\).

(a) Prove that \(E(t) = \frac{1}{2}m|\mathbf{v}(t)|^2 + mg\mathbf{r}(t) \cdot \mathbf{k}\) is a constant, independent of \(t\). (This is called “conservation of energy”.)

(b) Prove that the speed \(|\mathbf{v}|\) at the point \(\theta\) obeys \(|\mathbf{v}|^2 = 2gb(2\pi - \theta)\).

(c) Find the time it takes to reach \(\theta = 0\).