Theorem 2.4.2

\[ \int_C F = \nabla \phi \] is conservative and \( C \) is a curve that starts at \( P_0 \) and ends at \( P_1 \) then

\[ \int_C F \cdot dr = \phi(P_1) - \phi(P_0). \]

Proof. Parametrize \( C \):

\[ \vec{r}(t) = (x(t), y(t), z(t)), \quad a \leq t \leq b \]

\[ \int_C F \cdot dr = \int_a^b F(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}(t) \, dt = \int_a^b \nabla \phi(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}(t) \, dt \]

\[ = \int_a^b \left\{ \sum_{i=1}^3 \frac{\partial \phi}{\partial x}(x(t), y(t), z(t)) \frac{dx}{dt}(t) + \frac{\partial \phi}{\partial y}(x(t), y(t), z(t)) \frac{dy}{dt}(t) + \frac{\partial \phi}{\partial z}(x(t), y(t), z(t)) \frac{dz}{dt}(t) \right\} \, dt \]

\[ = \int_a^b \frac{d}{dt} \left[ \phi(x(t), y(t), z(t)) \right] \, dt \quad \text{(chain rule)} \]

\[ = \phi(\vec{r}(b)) - \phi(\vec{r}(a)) = \phi(P_1) - \phi(P_0) \quad \text{(FTOC)} \]

Consequence: \( \int_C F \) is conservative, \( \int_C F \cdot dr \) takes the same value for all curves \( C \) that start at any fixed \( P_0 \) and end at any fixed \( P_1 \). This called "path independence."