

Divergence Theorem and Variations

Theorem. *If V is a solid with surface ∂V*

$$\begin{aligned}\iint_{\partial V} \vec{v} \cdot \hat{\mathbf{n}} \, dS &= \iiint_V \vec{\nabla} \cdot \vec{v} \, dV \\ \iint_{\partial V} \psi \hat{\mathbf{n}} \, dS &= \iiint_V \vec{\nabla} \psi \, dV \\ \iint_{\partial V} \hat{\mathbf{n}} \times \vec{v} \, dS &= \iiint_V \vec{\nabla} \times \vec{v} \, dV\end{aligned}$$

where $\hat{\mathbf{n}}$ is the outward unit normal of ∂V .

Memory Aid. All three formulae can be combined into

$$\iint_{\partial V} \hat{\mathbf{n}} * \vec{v} \, dS = \iiint_V \vec{\nabla} * \vec{v} \, dV$$

where $*$ can be either \cdot , \times or nothing. When $*$ = \cdot or $*$ = \times , then $\vec{v} = \vec{v}$. When $*$ is nothing, $\vec{v} = v$.

Proof: The first formula is the divergence theorem and was proven in class.

To prove the second, assuming the first, apply the first with $\vec{v} = \psi \vec{a}$, where \vec{a} is any constant vector.

$$\begin{aligned}\iint_{\partial V} \psi \vec{a} \cdot \hat{\mathbf{n}} \, dS &= \iiint_V \vec{\nabla} \cdot (\psi \vec{a}) \, dV \\ &= \iiint_V [(\vec{\nabla} \psi) \cdot \vec{a} + \psi \vec{\nabla} \cdot \vec{a}] \, dV \\ &= \iiint_V (\vec{\nabla} \psi) \cdot \vec{a} \, dV\end{aligned}$$

To get the second line, we used vector identity # 8. To get the third line, we just used that \vec{a} is a constant, so that it is annihilated by all derivatives. Since \vec{a} is a constant, we can factor it out of both integrals, so

$$\begin{aligned}\vec{a} \cdot \iint_{\partial V} \psi \hat{\mathbf{n}} \, dS &= \vec{a} \cdot \iiint_V \vec{\nabla} \psi \, dV \\ \implies \vec{a} \cdot \left\{ \iint_{\partial V} \psi \hat{\mathbf{n}} \, dS - \iiint_V \vec{\nabla} \psi \, dV \right\} &= 0\end{aligned}$$

In particular, choosing $\vec{a} = \hat{i}, \hat{j}, \hat{k}$, we see that all three components of the vector $\iint_{\partial V} \psi \hat{\mathbf{n}} dS - \iiint_V \vec{\nabla} \psi dV$ are zero. So

$$\iint_{\partial V} \psi \hat{\mathbf{n}} dS - \iiint_V \vec{\nabla} \psi dV = 0$$

which is what we wanted show.

To prove the third, assuming the first, apply the first with \vec{v} replaced by $\vec{v} \times \vec{a}$, where \vec{a} is any constant vector.

$$\begin{aligned} \iint_{\partial V} \vec{v} \times \vec{a} \cdot \hat{\mathbf{n}} dS &= \iiint_V \vec{\nabla} \cdot (\vec{v} \times \vec{a}) dV \\ &= \iiint_V [(\vec{\nabla} \times \vec{v}) \cdot \vec{a} - \vec{v} \cdot \vec{\nabla} \times \vec{a}] dV \\ &= \iiint_V (\vec{\nabla} \times \vec{v}) \cdot \vec{a} dV \end{aligned}$$

To get the second line, we used vector identity # 9. To get the third line, we just used that \vec{a} is a constant, so that it is annihilated by all derivatives. For all vectors

$$\vec{v} \times \vec{a} \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \vec{v} \times \vec{a} = \hat{\mathbf{n}} \times \vec{v} \cdot \vec{a}$$

so

$$\begin{aligned} \vec{a} \cdot \iint_{\partial V} \hat{\mathbf{n}} \times \vec{v} dS &= \vec{a} \cdot \iiint_V \vec{\nabla} \times \vec{v} dV \\ \implies \vec{a} \cdot \left\{ \iint_{\partial V} \hat{\mathbf{n}} \times \vec{v} dS - \iiint_V \vec{\nabla} \times \vec{v} dV \right\} &= 0 \end{aligned}$$

In particular, choosing $\vec{a} = \hat{i}, \hat{j}, \hat{k}$, we see that all three components of the vector $\iint_{\partial V} \hat{\mathbf{n}} \times \vec{v} dS - \iiint_V \vec{\nabla} \times \vec{v} dV$ are zero. So

$$\iint_{\partial V} \hat{\mathbf{n}} \times \vec{v} dS - \iiint_V \vec{\nabla} \times \vec{v} dV = 0$$

which is what we wanted show. ■