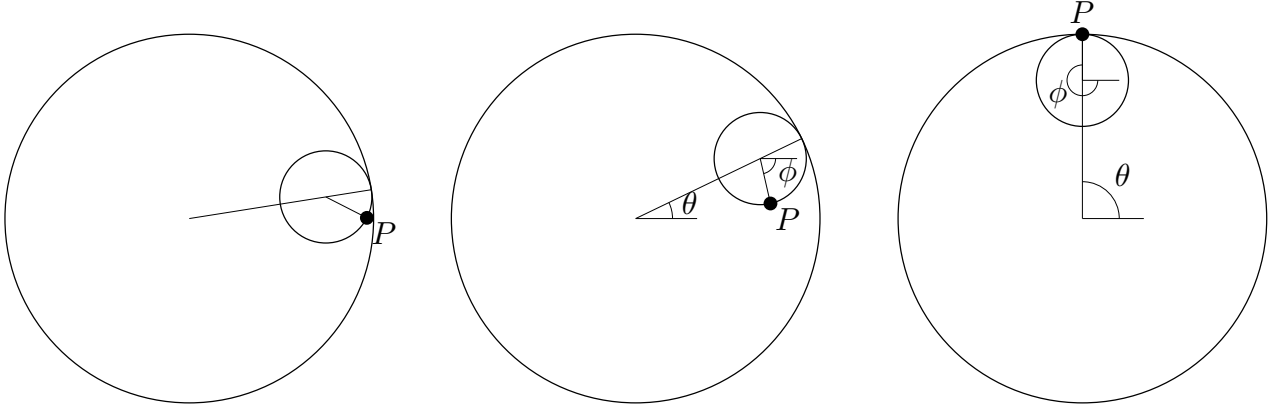


The Astroid

Imagine a ball of radius $a/4$ rolling around the inside of a circle of radius a . We shall now find the equation of the curve traced by a point P painted on the inner circle. You can find a java applet demonstrating this curve on the web at

<http://www.ugrad.math.ubc.ca/coursedoc/math100/notes/derivative/implicit.html>

Define the angles θ and ϕ as in the figure.



As θ runs from 0 to $\frac{\pi}{2}$, the point of contact between the two circles travels a distance $\frac{1}{4}(2\pi a)$, which is exactly the circumference of the inner circle. Hence if $\phi = 0$ for $\theta = 0$ (i.e. P starts on the x -axis), then for $\theta = \frac{\pi}{2}$, P is back in contact with the big circle at the north pole of both the inner and outer circles. That is, $\phi = -\frac{3\pi}{2}$ when $\theta = \frac{\pi}{2}$. So $\phi = -3\theta$ and P has coordinates

$$\frac{3}{4}a(\cos \theta, \sin \theta) + \frac{1}{4}a(\cos \phi, \sin \phi) = \frac{a}{4}(3 \cos \theta + \cos 3\theta, 3 \sin \theta - \sin 3\theta)$$

As

$$\begin{aligned} \cos 3\theta &= \cos \theta \cos 2\theta - \sin \theta \sin 2\theta \\ &= \cos \theta[\cos^2 \theta - \sin^2 \theta] - 2 \sin^2 \theta \cos \theta \\ &= \cos \theta[\cos^2 \theta - 3 \sin^2 \theta] \\ \sin 3\theta &= \sin \theta \cos 2\theta + \cos \theta \sin 2\theta \\ &= \sin \theta[\cos^2 \theta - \sin^2 \theta] + 2 \sin \theta \cos^2 \theta \\ &= \sin \theta[3 \cos^2 \theta - \sin^2 \theta] \end{aligned}$$

we have

$$\begin{aligned} 3 \cos \theta + \cos 3\theta &= \cos \theta[3 + \cos^2 \theta - 3 \sin^2 \theta] = 4 \cos^3 \theta \\ 3 \sin \theta - \sin 3\theta &= \sin \theta[3 - 3 \cos^2 \theta + \sin^2 \theta] = 4 \sin^3 \theta \end{aligned}$$

and the coordinates of P simplify to

$$x(\theta) = a \cos^3 \theta \quad y(\theta) = a \sin^3 \theta$$

As $x^{2/3} + y^{2/3} = a^{2/3} \cos^2 \theta + a^{2/3} \sin^2 \theta$, the path traced by P obeys the equation

$$\boxed{x^{2/3} + y^{2/3} = a^{2/3}}$$

There remains the danger that there could exist points (x, y) obeying the equation $x^{2/3} + y^{2/3} = a^{2/3}$ that are not of the form $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ for any θ . That is, there is a danger that the parametrized curve $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ covers only a portion of $x^{2/3} + y^{2/3} = a^{2/3}$. We now show that the parametrized curve $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ in fact covers all of $x^{2/3} + y^{2/3} = a^{2/3}$ as θ runs from 0 to 2π . First, observe that $x^{2/3} = (\sqrt[3]{x})^2 \geq 0$ and $y^{2/3} = (\sqrt[3]{y})^2 \geq 0$. Hence, if (x, y) obeys $x^{2/3} + y^{2/3} = a^{2/3}$, then necessarily $0 \leq x^{2/3} \leq a^{2/3}$ and so $-a \leq x \leq a$. As θ runs from 0 to 2π , $a \cos^3 \theta$ takes all values between $-a$ and a and hence takes all possible values of x . For each $x \in [-a, a]$, y takes two values, namely $\pm[a^{2/3} - x^{2/3}]^{3/2}$. If $x = a \cos^3 \theta_0 = a \cos^3(2\pi - \theta_0)$, the two corresponding values of y are precisely $a \sin^3 \theta_0$ and $-a \sin^3 \theta_0 = a \sin^3(2\pi - \theta_0)$.