

# Proofs that $\vec{a} \cdot \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \cdot \vec{c}$

## Proof One:

$$\begin{aligned}\vec{a} \cdot \vec{b} \times \vec{c} &= (a_1, a_2, a_3) \cdot (b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\ \vec{a} \times \vec{b} \cdot \vec{c} &= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \cdot (c_1, c_2, c_3) \\ &= a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3\end{aligned}$$

## Proof Two:

$$\begin{aligned}\vec{a} \cdot \vec{b} \times \vec{c} &= (a_1, a_2, a_3) \cdot \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \\ &= a_1 \det \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} - a_2 \det \begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix} + a_3 \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \\ &= \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \\ \vec{a} \times \vec{b} \cdot \vec{c} &= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \cdot (c_1, c_2, c_3) \\ &= c_1 \det \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} - c_2 \det \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} + c_3 \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \\ &= \det \begin{bmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\end{aligned}$$

Exchanging two rows in a determinant changes the sign of the determinant. Moving the top row of a  $3 \times 3$  determinant to the bottom row requires two exchanges of rows. So the two  $3 \times 3$  determinants are equal.