1. (a) Find the curvature of $y = e^x$ at $(0, 1)$.
(b) Find the equation of the circle best fitting $y = e^x$ at $(0, 1)$.

**Solution.** Parametrize the curve by $r(t) = t\hat{i} + e^t\hat{j}$. Then

$v(t) = \hat{i} + e^t\hat{j}$ \hspace{1cm} $v(0) = \hat{i} + \hat{j}$ \hspace{1cm} $\frac{dv}{dt}(0) = \sqrt{2}$ \hspace{1cm} $\hat{T}(0) = \frac{v(0)}{|v(0)|} = \frac{\hat{i} + \hat{j}}{\sqrt{2}}$

$a(t) = e^t\hat{j}$ \hspace{1cm} $a(0) = \hat{j}$

(a) Since

$v(0) \times a(0) = \hat{k} = \kappa(0) \left(\frac{dv}{dt}(0)\right)^3 \hat{B} = \kappa(0) 2^{3/2} \hat{B}$

we have $\kappa(0) = 2^{-3/2}$ and $\hat{B} = \hat{k}$.

(b) We have

$\hat{N}(0) = \hat{B}(0) \times \hat{T}(0) = \frac{1}{\sqrt{2}} \hat{k} \times (\hat{i} + \hat{j}) = \frac{1}{\sqrt{2}} (\hat{i} - \hat{j})$

so that the radius of curvature is $\frac{1}{\kappa(0)} = 2^{3/2}$ and centre of curvature is

$(0, 1) + \frac{1}{\kappa(0)} \hat{N}(0) = (0, 1) + 2^{3/2} 2^{-1/2} (-1, 1) = (-2, 3)$

and the equation of the osculating circle is $\left( x + 2 \right)^2 + \left( y - 3 \right)^2 = 8$.

2. The surface $z = x^2 + y^2$ is sliced by the plane $x = y$. The resulting curve is oriented from $(0, 0, 0)$ to $(1, 1, 2)$.
(a) Sketch the curve from $(0, 0, 0)$ to $(1, 1, 2)$.
(b) Sketch $\hat{T}$, $\hat{N}$ and $\hat{B}$ at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
(c) Find the torsion at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

**Solution.** (a), (b)
3. (a) For which value(s) of the constants $a, b$ is the vector field

$$\mathbf{F} = (2x \sin(\pi y) - e^z)\hat{i} + (ax^2 \cos(\pi y) - 3e^z)\hat{j} - (x + by)e^z\hat{k}$$

conservative?

(b) Let $\mathbf{F}$ be a conservative field from part (a). Find all functions $\phi$ for which $\mathbf{F} = \nabla \phi$.

(c) Let $\mathbf{F}$ be a conservative field from part (a). Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $C$ is the intersection of $y = x$ and $z = \ln(1 + x)$ from $(0, 0, 0)$ to $(1, 1, \ln 2)$.

(d) Evaluate $\oint_C \mathbf{G} \cdot d\mathbf{r}$ where

$$\mathbf{G} = (2x \sin(\pi y) - e^z)\hat{i} + (\pi x^2 \cos(\pi y) - 3e^z)\hat{j} - xe^z\hat{k}$$

and $C$ is the intersection of $y = x$ and $z = \ln(1 + x)$ from $(0, 0, 0)$ to $(1, 1, \ln 2)$.

**Solution.** (a) The field is conservative only if

$$\frac{\partial F_y}{\partial y} = \frac{\partial F_x}{\partial x}, \quad \frac{\partial F_z}{\partial z} = \frac{\partial F_y}{\partial y}$$

That is,

$$\frac{\partial}{\partial y} (2x \sin(\pi y) - e^z) = \frac{\partial}{\partial x} (ax^2 \cos(\pi y) - 3e^z) \iff 2\pi x \cos(\pi y) = 2ax \cos(\pi y)$$

$$\frac{\partial}{\partial x} (2x \sin(\pi y) - e^z) = -\frac{\partial}{\partial x} (x + by) e^z \iff -e^z = -e^z$$

$$\frac{\partial}{\partial z} (ax^2 \cos(\pi y) - 3e^z) = -\frac{\partial}{\partial y} (x + by) e^z \iff -3e^z = -3e^z$$

Hence only $a = \pi, \ b = 3$ works.

(b) When $a = \pi, \ b = 3$

$$\mathbf{F} = (2x \sin(\pi y) - e^z)\hat{i} + (\pi x^2 \cos(\pi y) - 3e^z)\hat{j} - (x + 3y)e^z\hat{k}$$

$$= \nabla (x^2 \sin(\pi y) - xe^z - 3ye^z + C)$$

so $\phi(x, y, z) = x^2 \sin(\pi y) - xe^z - 3ye^z + C$ for any constant $C$. Here $\phi$ was guessed. Alternatively, it can be found by solving

$$\frac{\partial \phi}{\partial x}(x, y, z) = 2x \sin(\pi y) - e^z$$

$$\frac{\partial \phi}{\partial y}(x, y, z) = \pi x^2 \cos(\pi y) - 3e^z$$

$$\frac{\partial \phi}{\partial z}(x, y, z) = -(x + 3y)e^z$$

Integrating the first of these equations gives

$$\phi(x, y, z) = x^2 \sin(\pi y) - xe^z + g(y, z)$$

Substituting this into the second equation gives

$$\pi x^2 \cos(\pi y) + \frac{\partial g}{\partial y}(y, z) = \pi x^2 \cos(\pi y) - 3e^z \quad \text{or} \quad \frac{\partial g}{\partial y}(y, z) = -3e^z$$

which forces

$$g(y, z) = -3ye^z + h(z)$$

Finally, substituting $\phi(x, y, z) = x^2 \sin(\pi y) - xe^z - 3ye^z + h(z)$ into the last equation gives

$$-xe^z - 3ye^z + h'(z) = -(x + 3y)e^z \quad \text{or} \quad h'(z) = 0$$

So $h(x) = C$ and hence $\phi(x, y, z) = x^2 \sin(\pi y) - xe^z - 3ye^z + C$ for any constant $C$. 


(c) By part (b),
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \phi(1, 1, \ln 2) - \phi(0, 0, 0) = 8 \]

(d) Observe that \( \mathbf{G} = \mathbf{F} + 3ye^z \hat{k} \), with \( \mathbf{F} \) evaluated with \( a = \pi, \ b = 3 \). Hence
\[ \int_C \mathbf{G} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C 3ye^z \hat{k} \cdot d\mathbf{r} = -8 + \int_C 3ye^z \hat{k} \cdot d\mathbf{r} \]
To evaluate the remaining integral, parametrize the curve by \( \mathbf{r}(t) = t\mathbf{i} + ty + \ln(1 + t)\hat{k} \) with \( 0 \leq t \leq 1 \). Then \( \mathbf{r}'(t) = \mathbf{i} + \frac{t}{1 + t^2} \hat{k} \) and \( 3ye^z = 3t(1 + t)\hat{k} \) so that \( 3ye^z \hat{k} \cdot d\mathbf{r} = 3t \, dt \). Subbing in
\[ \int_C \mathbf{G} \cdot d\mathbf{r} = -8 + \int_0^1 3t \, dt = -8 + \frac{3}{2} = \frac{-13}{2} \]

4. Let the thin shell \( S \) consist of the part of the surface \( z^2 = 2xy \) with \( x \geq 1, \ y \geq 1 \) and \( 0 \leq z \leq 2 \). Find the mass of \( S \) if it has surface density given by \( \rho(x, y, z) = 3z \) kg per unit area.

**Solution.** The surface is \( z = f(x, y) \) with \( f(x, y) = \sqrt{2xy} \). Since \( f_x = \frac{\sqrt{2}}{2x} \) and \( f_y = \frac{\sqrt{2}}{2y} \),
\[ dS = \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy = \sqrt{1 + \frac{2}{2x} + \frac{2}{2y}} \, dx \, dy \]
The domain of integration is
\[ \{ (x, y) \mid x \geq 1, \ y \geq 1, \ 2xy = z^2 \leq 4 \} = \{ (x, y) \mid x \geq 1, \ y \geq 1, \ xy \leq 2 \} \]
\[ = \{ (x, y) \mid 1 \leq x \leq 2, \ 1 \leq y \leq \frac{2}{x} \} \]

So the mass is
\[ \iint_S \rho(x, y, z) \, dS = \int_1^2 dx \int_1^{2/x} \left[ f(x, y) \sqrt{1 + \frac{y}{2x} + \frac{x}{2y}} \right] \, dy = \int_1^2 dx \int_1^{2/x} \left[ \sqrt{2xy} \sqrt{1 + \frac{y}{2x} + \frac{x}{2y}} \right] \, dy \]
\[ = \int_1^2 dx \int_1^{2/x} \left[ \sqrt{2xy(y^2 + x^2)} \right] \, dy \]
\[ = \int_1^2 dx \left[ xy + \frac{1}{2} y^2 \right]_{y=1}^{y=2/x} = 3 \int_1^2 dx \left[ 2 + \frac{2}{2x} - x - \frac{1}{2} \right] = 3 \left[ \frac{3}{2} - \frac{2}{2} + \frac{2}{2} - 2 + \frac{1}{2} \right] = 3 \text{ kg} \]

5. Let \( \mathbf{F} = (x^2 + y^2 + z^2)\mathbf{i} + (e^{x^2 + y^2})\mathbf{j} + (3 + x + z)\hat{k} \) and let \( S \) be the part of the surface \( x^2 + y^2 + z^2 = 2az + 3a^2 \) having \( z \geq 0 \), oriented with normal pointing away from the origin. Here \( a > 0 \) is a constant. Compute the flux of \( \mathbf{F} \) through \( S \).
Solution. Note that, since \( z^2 - 2az = (z - a)^2 - a^2 \),

\[
S = \left\{ (x, y, z) \mid x^2 + y^2 + (z - a)^2 = 4a^2, \ z \geq 0 \right\}
\]

Let \( V \) be the solid

\[
V = \left\{ (x, y, z) \mid x^2 + y^2 + (z - a)^2 \leq 4a^2, \ z \geq 0 \right\}
\]

Then the surface of \( V \) (with outward normal) is the union of \( S \) (with normal pointing away from the origin) and the disk

\[
B = \left\{ (x, y, 0) \mid x^2 + y^2 \leq 3a^2 \right\}
\]

with normal \(-\hat{k}\). Hence, by the Divergence Theorem

\[
\iiint_S F \cdot \hat{n} \, dS = \iiint_V \nabla \cdot F \, dV - \iiint_B F \cdot (\hat{k}) \, dS
\]

Both \( V \) and \( B \) are invariant under \( x \to -x \) and under \( y \to -y \), so \( \iiint_V x \, dV = \iiint_V y \, dV = \iiint_B x \, dS = 0 \) and

\[
\iiint_S F \cdot \hat{n} \, dS = \iiint_V \, dV + 3 \iiint_B \, dS
\]

To evaluate the integral over \( V \), we note that \( z \) runs from 0 to \( 3a \) and that the cross section of \( V \) with fixed \( z \) is the circular disk \( x^2 + y^2 \leq 4a^2 - (z - a)^2 = 3a^2 - 2az - z^2 \), which has area \( \pi \left( \sqrt{3a^2 + 2az - z^2} \right)^2 \).

So

\[
\iiint_S F \cdot \hat{n} \, dS = \int_0^{3a} \pi \left( \sqrt{3a^2 + 2az - z^2} \right)^2 \, dz + 3 \text{Area}(B)
\]

\[
= \pi \int_0^{3a} (3a^2 + 2az - z^2) \, dz + 3\pi(3a^2)
\]

\[
= \pi \left( 3a^2 \times 3a + 2a \times 2a^2 - \frac{27a^3}{3} + 9\pi a^2 \right) = 9\pi a^3 + 9\pi a^2
\]

6. Let \( C \) be the counterclockwise boundary of the rectangle with vertices \((1, 0), (3, 0), (3, 1)\) and \((1, 1)\). Evaluate

\[
\oint_C (3y^2 + 2xe^{y^2}) \, dx + (2yxe^{y^2}) \, dy
\]

Solution. Let’s use Green’s theorem. The rectangle, which we shall denote \( R \), is

\[
R = \left\{ (x, y) \mid 1 \leq x \leq 3, \ 0 \leq y \leq 1 \right\}
\]

So Green’s theorem gives

\[
\oint_C (3y^2 + 2xe^{y^2}) \, dx + (2yxe^{y^2}) \, dy = \iint_R \left[ \frac{\partial}{\partial x} (2yxe^{y^2}) - \frac{\partial}{\partial y} (3y^2 + 2xe^{y^2}) \right] \, dxdy
\]

\[
= \iint_R \left[ 4xye^{y^2} - 6y - 4yxe^{y^2} \right] \, dxdy
\]

\[
= -6 \int_1^3 \, dx \int_0^1 \, dy \ y = -6 \int_1^3 \, dx \ \frac{1}{2}
\]

\[
= -6
\]
7. Let $S$ be the part of the half cone
\[ z = \sqrt{x^2 + y^2}, \quad y \geq 0, \]
that lies below the plane $z = 1$.
(a) Find a parametrization for $S$.
(b) Calculate the flux of the velocity field
\[ \mathbf{v} = x \mathbf{i} + y \mathbf{j} - 2z \mathbf{k} \]
downward through $S$.
(c) A vector field $\mathbf{F}$ has curl $\nabla \times \mathbf{F} = x \mathbf{i} + y \mathbf{j} - 2z \mathbf{k}$. On the $xz$-plane, the vector field $\mathbf{F}$ is constant with $\mathbf{F}(x, 0, z) = \mathbf{j}$. Given this information, calculate
\[ \int_C \mathbf{F} \cdot d\mathbf{r}, \]
where $C$ is the half circle \(x^2 + y^2 = 1, \ z = 1, \ y \geq 0\) oriented from $(-1, 0, 1)$ to $(1, 0, 1)$.

**Solution.** (a) We parametrize $S$ in cylindrical coordinates:
\[ \mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + r \mathbf{k} \quad \text{with} \quad 0 \leq r \leq 1, \ 0 \leq \theta \leq \pi \]
(b) We compute
\[ \frac{\partial \mathbf{r}}{\partial r} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} + \mathbf{k} \]
\[ \frac{\partial \mathbf{r}}{\partial \theta} = -r \sin \theta \mathbf{i} + r \cos \theta \mathbf{j} \]
\[ \hat{n} dS = \pm \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \, dr \, d\theta = \pm \left( -r \cos \theta \mathbf{i} - r \sin \theta \mathbf{j} + r \mathbf{k} \right) \, dr \, d\theta \]
To calculate the downward flux, we use the minus sign. We find
\[
\int_S \mathbf{v} \cdot \hat{n} \, dS = \int_0^\pi d\theta \int_0^1 dr \ (r \cos \theta, r \sin \theta, -2r) \cdot (r \cos \theta, r \sin \theta, -r)
\]  
\[
= \int_0^\pi d\theta \int_0^1 dr \ 3r^2 = \pi r^3 \bigg|_{r=0}^1 = \pi
\]
(c) **Solution 1:** Let $P$ be the path along line segments from $(1, 0, 1)$ to $(0, 0, 0)$ and from $(0, 0, 0)$ to $(-1, 0, 1)$. Here is a sketch. $P$ is in blue.
Then
\[ \int_{C} \mathbf{F} \cdot d\mathbf{r} + \int_{P} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} \nabla \times \mathbf{F} \cdot \hat{n} dS \]
by Stokes' Theorem. Along \( P \), the vector field \( \mathbf{F} \) is orthogonal to the curve so that \( \int_{P} \mathbf{F} \cdot d\mathbf{r} = 0 \). Note that \( \nabla \times \mathbf{F} \) is the vector field \( \mathbf{v} \) from part (b). Thus
\[ \int_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} \mathbf{v} \cdot \hat{n} dS = \pi \]

(c) **Solution 2:** Let \( \mathcal{L} \) be the line segment from \((1,0,1)\) to \((-1,0,1)\) and let
\[ \mathcal{R} = \{ (x,y,z) | x^2 + y^2 \leq 1, y \geq 0, z = 1 \} \]
Here is a sketch. \( \mathcal{L} \) is in blue and \( \mathcal{R} \) is shaded.

Then
\[ \int_{C} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{L}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{R}} \nabla \times \mathbf{F} \cdot (-\hat{k}) dS \]
by Stokes' Theorem. Along \( \mathcal{L} \), the vector field \( \mathbf{F} = \hat{j} \) is orthogonal to the curve (which has direction \(-\hat{i}\) so that \( \int_{\mathcal{L}} \mathbf{F} \cdot d\mathbf{r} = 0 \). Note that \( \nabla \times \mathbf{F} \) is the vector field \( \mathbf{v} \) from part (b). Thus
\[ \int_{C} \mathbf{F} \cdot d\mathbf{r} = -\iint_{\mathcal{R}} \mathbf{v} \cdot \hat{k} dS = \iint_{\mathcal{R}} 2z dS = 2 \iint_{\mathcal{R}} dS = 2 \text{Area}(\mathcal{R}) = \pi \]

8. A region \( R \) is bounded by a simple closed curve \( C \). The curve \( C \) is oriented such that \( R \) lies to the left of \( C \) when walking along \( C \) in the direction of \( C \). Determine whether or not each of the following expressions is equal to the area of \( R \). You must justify your conclusions.
(a) \( \frac{1}{2} \int_{C} -y \, dx + x \, dy \)
(b) \( \frac{1}{2} \int_{C} -x \, dx + y \, dy \)
(c) \( \int_{C} y \, dx \)
(d) \( \int_{C} 3y \, dx + 4x \, dy \)
**Solution.** We apply Green’s Theorem:
\[ \int_{C} F_{1} \, dx + F_{2} \, dy = \iint_{R} \left( \frac{\partial F_{2}}{\partial x} - \frac{\partial F_{1}}{\partial y} \right) \, dxdy \]

(a) \( \frac{1}{2} \int_{C} -y \, dx + x \, dy = \frac{1}{2} \iint_{R} \{1 - (-1)\} \, dx \, dy = \text{Area}(R) \)
(b) \( \frac{1}{2} \int_C -x \, dx + y \, dy = \frac{1}{2} \iint_R 0 \, dx \, dy = 0 \neq \text{Area}(R) \)

(c) \( \int_C y \, dx = \iint_R \{ -1 \} \, dx \, dy = -\text{Area}(R) \neq \text{Area}(R) \)

(d) \( \int_C 3y \, dx + 4x \, dy = \iint_R \{ 4 - 3 \} \, dx \, dy = \text{Area}(R) \)

9. Say whether each of the following statements is true or false and explain why.
(a) A moving particle has velocity and acceleration vectors that satisfy \(|v| = 1\) and \(|a| = 1\) at all times. Then the curvature of this particle’s path is a constant.

(b) If \( F \) is any smooth vector field defined in \( \mathbb{R}^3 \) and if \( S \) is any sphere, then
\[
\iint_S \nabla \times F \cdot \hat{n} \, dS = 0
\]
Here \( \hat{n} \) is the outward normal to \( S \).

(c) If \( F \) and \( G \) are smooth vector fields in \( \mathbb{R}^3 \) and if \( \oint_C F \cdot dr = \oint_C G \cdot dr \) for every circle \( C \), then \( F = G \).

Solution. (a) True. Since \( v = |v| = 1 \) is constant, we have
\[
a = \frac{dv}{dt} \mathbf{T} + v^2 \kappa \hat{N} = 0 \mathbf{T} + \kappa \hat{N}.
\]
Thus \( 1 = |a| = \kappa |\hat{N}| \), i.e., \( \kappa = 1 \).

(b) True. By the divergence theorem, if \( V \) is the solid bounded by \( S \),
\[
\iint_S \nabla \times F \cdot \hat{n} \, dS = \iiint_V \nabla \cdot (\nabla \times F) \, dV = 0
\]
since \( \nabla \cdot (\nabla \times F) = 0 \).

(c) False. If \( F = 0 \) and \( G \) is any nonzero, conservative field, like \( G = 2x \hat{i} = \nabla(x^2) \), then
\[
\oint_C F \cdot dr = \oint_C G \cdot dr = 0
\]
for every closed curve \( C \).