MATHEMATICS 317 Practice Midterm 1a

1. (a) Calculate the work done by the force field \( F = (-ay, bx, -cz) \) on a particle that moves on a path \( C \), where \( C \) is composed of
   i. a straight line from \((0, -1, 0)\) to \((0, 1, 0)\),
   ii. straight lines from \((0, -1, 0)\) to \((1, -1, 0)\) to \((0, 1, 0)\).
(b) What is the condition on \( a \), \( b \), and \( c \) for \( F \) to be conservative? Hint: you may use part (a). (Be careful, you have to prove that the condition you find is necessary AND sufficient.)
   (c) For \( a = 2 \), \( b = -2 \) and \( c = 1 \), evaluate the line integral of \( F \) along the curve \( C \) given by \( \mathbf{r}(t) = \left( \ln(1 + t^5), e^{t^7}, t \right) \) with \( 0 \leq t \leq 1 \).

2. Compute the line integral of the vector field \( F(x, y) = (ye^{xy}, xe^{xy} + x) \) along the curve \( x(t) = \cos(t) \), \( y(t) = 4\sin(t) \) where \( 0 \leq t \leq 2\pi \). Hint: decompose \( F \) as a sum of two vectors fields, and justify that one of them is conservative.

3. Calculate the arc length of the curve parameterized by
   \[
   x(t) = \frac{1}{2} t + \frac{1}{2} \cos t \sin t, \quad y(t) = \frac{1}{2} \sin^2 t; \quad 0 \leq t \leq \pi.
   \]
   Suggestion: you may use the formulas \( \sin(2t) = 2\sin(t)\cos(t) \) and then \( 1 + \cos(2t) = 2\cos^2(t) \) to simplify the calculations.

4. Consider the curve \( C \) given by \( \mathbf{r}(t) = e^{-t}\cos(t)\hat{i} + e^{-t}\sin(t)\hat{j} + \sqrt{2}e^{-t}\hat{k} \).
   (a) Let \( L(t) \) denote the arc length of the curve from the point \((1, 0, 2)\) to the point with parameter \( t \). Compute \( \lim_{t \to +\infty} L(t) \).
   (b) Compute the unit tangent vector \( \hat{T}(t) \).
   (c) Compute the unit normal vector \( \hat{N}(t) \) and check that \( \hat{N}(0) = (-\sqrt{2}/2, -\sqrt{2}/2, 0) \).
   (d) Give the coordinates of the center of the osculating circle at parameter \( t = 0 \). We recall that the curvature at parameter \( t \) is given by \( \kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \) and that the osculating circle at parameter \( t \) has radius \( 1/\kappa(t) \).
   (e) Give the equation of the osculating plane at parameter \( t = 0 \).
   (f) Is \( C \) a plane curve?