MATHEMATICS 317 December 2017 Final Exam

1. Let
$$\mathbf{r} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}, \qquad r = |\mathbf{r}|$$
   (a) Compute a where $\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -r^a\,\mathbf{r}$.
   (b) Compute a where $\boldsymbol{\nabla}\cdot\big(r\,\mathbf{r}\big) = ar$.
   (c) Compute a where $\boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}(r^3)\big) = ar$.

2. Let $\mathbf{r}(t) = \left(\frac{1}{3}t^3, \frac{1}{2}t^2, \frac{1}{2}t\right)$, $t \geq 0$. Compute $s(t)$, the arclength of the curve at time t.

3. A particle of mass
$$m = 2$$
   is acted on by a force
$$\mathbf{F} = \big(4t\,,\,6t^2\,,\,-4t\big)$$
   At $t = 0$, the particle has velocity zero and is located at the point $(1, 2, 3)$.
   (a) Find the velocity vector $\mathbf{v}(t)$ for $t \geq 0$.
   (b) Find the position vector $\mathbf{r}(t)$ for $t \geq 0$.
   (c) Find $\kappa(t)$ the curvature of the path traversed by the particle for $t \geq 0$.
   (d) Find the work done by the force on the particle from $t = 0$ to $t = T$.

4. Let
$$\mathbf{F} = \left(\frac{2z}{1+y} + \sin(x^2)\,,\,\frac{3z}{1+x} + \sin(y^2)\,,\,5(x+1)(y+2)\right)$$
   Let C be the oriented curve consisting of four line segments from $(0, 0, 0)$ to $(2, 0, 0)$, from $(2, 0, 0)$ to $(0, 0, 2)$, from $(0, 0, 2)$ to $(0, 3, 0)$, and from $(0, 3, 0)$ to $(0, 0, 0)$.
   (a) Draw a picture of C. Clearly indicate the orientation on each line segment.
   (b) Compute the work integral $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$.

5. Recall that if $\hat{\mathbf{T}}$ is the unit tangent vector to an oriented curve with arclength parameter s, then the curvature κ and the principle normal vector $\hat{\mathbf{N}}$ are defined by the equation
$$\frac{\mathrm{d}\hat{\mathbf{T}}}{\mathrm{d}s} = \kappa\,\hat{\mathbf{N}}$$
   Moreover, the torsion τ and the binormal vector $\hat{\mathbf{B}}$ are defined by the equations
$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}, \qquad \frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} = -\tau\,\hat{\mathbf{N}}$$
   Show that
$$\frac{\mathrm{d}\hat{\mathbf{N}}}{\mathrm{d}s} = -\kappa\,\hat{\mathbf{T}} + \tau\,\hat{\mathbf{B}}$$

6. Let a, b, c be constants and let

$$\mathbf{F} = \left(2bx^2y^2 - axz\,,\ 2ax^3y - 3cz^2\,,\ 2byz + x^2\right)$$

   (a) Compute $\boldsymbol{\nabla} \times \mathbf{F}$.

   (b) Find the values of a, b, c which make $\mathbf{F}$ conservative.

   (c) Using the values of a, b, c from the previous part, find a potential function for $\mathbf{F}$.

   (d) Using the values of a, b, c from the previous part, evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where $\mathbf{r}(t) = \left(\cos(t)\,,\ 2\sin(t)\,,\ \frac{2t}{\pi} + 1\right)$ for $0 \le t \le \frac{\pi}{2}$.

7. (a) Evaluate

$$\int_C \sqrt{1 + x^3}\,\mathrm{d}x + \left(2xy^2 + y^2\right)\mathrm{d}y$$

   where C is the unit circle $x^2 + y^2 = 1$, oriented counterclockwise.

   (b) Evaluate

$$\int_C \sqrt{1 + x^3}\,\mathrm{d}x + \left(2xy^2 + y^2\right)\mathrm{d}y$$

   where C is now the part of the unit circle $x^2 + y^2 = 1$, with $x \ge 0$, still oriented counterclockwise.

8. Let S be the part of the sphere $x^2 + y^2 + z^2 = 2$ where $y \ge 1$, oriented away from the origin.

   (a) Compute

$$\iint_S y^3\,\mathrm{d}S$$

   (b) Compute

$$\iint_S \left(xy\,\hat{\boldsymbol{\imath}} + xz\,\hat{\boldsymbol{\jmath}} + zy\,\hat{\mathbf{k}}\right) \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

9. Let S be the sphere $x^2 + y^2 + z^2 = 3$ oriented inward. Compute the flux integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

   where

$$\mathbf{F} = \left(xy^2 + y^4z^6\,,\ yz^2 + x^4z\,,\ zx^2 + xy^4\right)$$