MATHEMATICS 317 April 2015 Final Exam

1. (a) Compute and simplify $\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r}}{r}\right)$ for $\mathbf{r} = (x, y, z)$ and $r = |(x, y, z)|$. Express your answer in terms of r.

   (b) Compute $\boldsymbol{\nabla} \times \left(yz\,\hat{\boldsymbol{\imath}} + 2xz\,\hat{\boldsymbol{\jmath}} + e^{xy}\,\hat{\mathbf{k}}\right)$.

   (c) Find the length of the curve $\mathbf{r}(t) = \left(1, \frac{t^2}{2}, \frac{t^3}{3}\right)$ for $0 \le t \le 1$.

   (d) Find the principal unit normal vector to $\mathbf{r}(t) = \cos(t)\,\hat{\boldsymbol{\imath}} + \sin(t)\,\hat{\boldsymbol{\jmath}} + t\,\hat{\mathbf{k}}$ at $t = \pi/4$.

   (e) Find the curvature of $\mathbf{r}(t) = \cos(t)\,\hat{\boldsymbol{\imath}} + \sin(t)\,\hat{\boldsymbol{\jmath}} + t\,\hat{\mathbf{k}}$ at $t = \pi/4$.

2. Let S be the surface obtained by revolving the curve $z = e^y$, $0 \le y \le 1$, around the y–axis where the orientation of S is where $\hat{\mathbf{n}}$ points toward the y–axis.

   (a) Draw a picture of S and find a parameterization of S.

   (b) Compute the integral $\iint_S e^y \,\mathrm{d}S$.

   (c) Compute the flux integral $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \,\mathrm{d}S$ where $\mathbf{F} = (x, 0, z)$.

3. Let C be the parameterized curve given by

$$\mathbf{r}(t) = \left(\cos t, \sin t, t\right), \qquad 0 \le t \le \frac{\pi}{2}$$

   and let

$$\mathbf{F} = \left(e^{yz}\,,\, xze^{yz} + ze^y\,,\, xye^{yz} + e^y\right)$$

   (a) Compute and simplify $\boldsymbol{\nabla} \times \mathbf{F}$.

   (b) Compute the work integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

4. (a) Use Green's Theorem to evaluate the line integral

$$\int_C \frac{-y}{x^2 + y^2}\mathrm{d}x + \frac{x}{x^2 + y^2}\mathrm{d}y$$

   where C is the arc of the parabola $y = \frac{1}{4}x^2 + 1$ from $(-2, 2)$ to $(2, 2)$.
   Hint: Green's theorem must be applied to a closed curve; note that the curve C is not closed. You may use the fact that $\int \frac{\mathrm{d}t}{1+t^2} = \arctan(t) + C$.

   (b) Use Green's Theorem to evaluate the line integral

$$\int_C \frac{-y}{x^2 + y^2}\mathrm{d}x + \frac{x}{x^2 + y^2}\mathrm{d}y$$

   where C is the arc of the parabola $y = x^2 - 2$ from $(-2, 2)$ to $(2, 2)$.
   Hint: Consider carefully the point $(0, 0)$ in your analysis of the situation.

(c) Is the vector field

$$\mathbf{F} = \frac{-y}{x^2+y^2}\hat{\boldsymbol{\imath}} + \frac{x}{x^2+y^2}\hat{\boldsymbol{\jmath}}$$

conservative? Provide a reason for your answer based on your answers to the previous parts of this question.

5. Consider the curve C that is the intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$, and suppose C is oriented so that it is traversed *clockwise* as seen from above.

   Let $\mathbf{F}(x, y, z) = \big(x^3 + 2y\,,\, \sin(y) + z\,,\, x + \sin(z^2)\big)$.

   Use Stokes' Theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

6. Let E be the solid region between the plane $z = 4$ and the paraboloid $z = x^2 + y^2$. Let

$$\mathbf{F} = \Big(-\frac{1}{3}x^3 + e^{z^2}\Big)\hat{\boldsymbol{\imath}} + \Big(-\frac{1}{3}y^3 + x\tan z\Big)\hat{\boldsymbol{\jmath}} + 4z\hat{\mathbf{k}}$$

   (a) Compute the flux of $\mathbf{F}$ outward through the boundary of E.

   (b) Let S be the part of the paraboloid $z = x^2 + y^2$ lying below the $z = 4$ plane oriented so that $\hat{\mathbf{n}}$ has a positive $\hat{\mathbf{k}}$ component. Compute the flux of $\mathbf{F}$ through S.

7. A particle moves along a curve with position vector given by

$$\mathbf{r}(t) = \big(t + 2\,,\, 1 - t\,,\, t^2/2\big)$$

   for $-\infty < t < \infty$.

   (a) Find the velocity as a function of t.

   (b) Find the speed as a function of t.

   (c) Find the acceleration as a function of t.

   (d) Find the curvature as a function of t.

   (e) Recall that the decomposition of the acceleration into tangential and normal components is given by the formula

$$\mathbf{r}''(t) = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\hat{\mathbf{T}}(t) + \kappa(t)\Big(\frac{\mathrm{d}s}{\mathrm{d}t}\Big)^2\hat{\mathbf{N}}(t)$$

   Use this formula and your answers to the previous parts of this question to find $\hat{\mathbf{N}}(t)$, the principal unit normal vector, as a function of t.

   (f) Find an equation for the osculating plane at the point corresponding to $t = 0$.

   (g) Find the centre of the osculating circle at the point corresponding to $t = 0$.

8. Consider the following surfaces

- S_1 is the hemisphere given by the equation $x^2 + y^2 + z^2 = 4$ with $z \geq 0$.
- S_2 is the cylinder given by the equation $x^2 + y^2 = 1$.
- S_3 is the cone given by the equation $z^2 = x^2 + y^2$ with $z \geq 0$.

Consider the following parameterizations:

A. $\mathbf{r}(\theta, \phi) = \left(\sqrt{4}\cos\theta\sin\phi\,,\ \sqrt{4}\sin\theta\sin\phi\,,\ \sqrt{4}\cos\phi\right), \qquad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi/6$

B. $\mathbf{r}(\theta, \phi) = \left(\sqrt{4}\cos\theta\sin\phi\,,\ \sqrt{4}\sin\theta\sin\phi\,,\ \sqrt{4}\cos\phi\right), \qquad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi/4$

C. $\mathbf{r}(\theta, \phi) = \left(\sqrt{4}\cos\theta\sin\phi\,,\ \sqrt{4}\sin\theta\sin\phi\,,\ \sqrt{4}\cos\phi\right), \qquad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi/3$

D. $\mathbf{r}(\theta, z) = \left(\sqrt{4 - z^2}\cos\theta\,,\ \sqrt{4 - z^2}\sin\theta\,,\ z\right) \qquad 0 \leq \theta \leq 2\pi, \quad 1 \leq z \leq 2$

E. $\mathbf{r}(\theta, z) = \left(\sqrt{4 - z^2}\cos\theta\,,\ \sqrt{4 - z^2}\sin\theta\,,\ z\right) \qquad 0 \leq \theta \leq 2\pi, \quad \sqrt{2} \leq z \leq 2$

F. $\mathbf{r}(\theta, z) = \left(\sqrt{4 - z^2}\cos\theta\,,\ \sqrt{4 - z^2}\sin\theta\,,\ z\right) \qquad 0 \leq \theta \leq 2\pi, \quad \sqrt{3} \leq z \leq 2$

G. $\mathbf{r}(\theta, z) = \left(z\cos\theta\,,\ z\sin\theta\,,\ z\right) \qquad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 1$

H. $\mathbf{r}(\theta, z) = \left(z\cos\theta\,,\ z\sin\theta\,,\ z\right) \qquad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq \sqrt{2}$

I. $\mathbf{r}(\theta, z) = \left(z\cos\theta\,,\ z\sin\theta\,,\ z\right) \qquad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq \sqrt{3}$

J. $\mathbf{r}(x, y) = \left(x\,,\ y\,,\ \sqrt{x^2 + y2}\right) \qquad x^2 + y^2 \leq 1$

K. $\mathbf{r}(x, y) = \left(x\,,\ y\,,\ \sqrt{x^2 + y2}\right) \qquad x^2 + y^2 \leq \sqrt{2}$

L. $\mathbf{r}(x, y) = \left(x\,,\ y\,,\ \sqrt{x^2 + y2}\right) \qquad x^2 + y^2 \leq 2$

For each of the following, choose from above all of the valid parameterization of each of the given surfaces. Note that there may be one or more valid parameterization for each surface, and not necessarily all of the above parameterizations will be used.

(a) The part of S_1 contained inside S_2:

(b) The part of S_1 contained inside S_3:

(c) The part of S_3 contained inside S_2:

(d) The part of S_3 contained inside S_1: