MATHEMATICS 317 December 2012 Final Exam

1. (a) Consider the parametrized space curve

$$\mathbf{r}(t) = (\cos(t), \sin(t), t^2)$$

Find a parametric form for the tangent line at the point corresponding to $t = \pi$.

(b) Find the tangential component $a_T(t)$ of acceleration, as a function of t, for the parametrized space curve of (a).

2. (a) Let

$$\mathbf{r}(t) = (2\sin^3 t, 2\cos^3 t, 3\sin t\cos t)$$

Find the unit tangent vector to this parametrized curve at $t = \pi/3$, pointing in the direction of increasing t.

(b) Reparametrize the vector function $\mathbf{r}(t)$ from (a) with respect to arc length measured from the point $t = 0$ in the direction of increasing t.

3. (a) Consider the vector field $\mathbf{F} = (3y, x - 1)$ in $\mathbb{R}^2$. Compute the line integral

$$\int_L \mathbf{F} \cdot d\mathbf{r}$$

where L is the line segment from $(1, 1)$ to $(2, 2)$.

(b) Find an oriented path C from $(2, 2)$ to $(1, 1)$ such that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 4$$

where $\mathbf{F}$ is the vector field from (a).

4. (a) Find the curl of the vector field $\mathbf{F} = \big(2 + x^2 + z\ ,\ 0\ ,\ 3 + x^2 z\big)$.

(b) Let C be the curve in $\mathbb{R}^3$ from the point $(0, 0, 0)$ to the point $(2, 0, 0)$, consisting of three consecutive line segments connecting the points $(0, 0, 0)$ to $(0, 0, 3)$, $(0, 0, 3)$ to $(0, 1, 0)$, and $(0, 1, 0)$ to $(2, 0, 0)$. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}$ is the vector field from (a).

5. (a) Consider the surface S given by the equation

$$x^2 + z^2 = \cos^2 y$$

Find an equation for the tangent plane to S at the point $\left(\frac{1}{2}, \frac{\pi}{4}, \frac{1}{2}\right)$.

(b) Compute the integral

$$\iint_S \sin y \, \mathrm{d}S$$

where S is the part of the surface from (a) lying between the planes $y = 0$ and $y = \frac{1}{2}\pi$.

6. (a) Let S be the bucket shaped surface consisting of the cylindrical surface $y^2 + z^2 = 9$ between $x = 0$ and $x = 5$, and the disc inside the yz–plane of radius 3 centered at the origin. (The bucket S has a bottom, but no lid.) Orient S in such a way that the unit normal points outward. Compute the flux of the vector field $\boldsymbol{\nabla} \times \mathbf{G}$ through S, where $\mathbf{G} = (x, -z, y)$.

   (b) Compute the flux of the vector field $\mathbf{F} = (2 + z, xz^2, x \cos y)$ through S, where S is as in (a).

7. (a) Find the divergence of the vector field $\mathbf{F} = (z + \sin y, zy, \sin x \cos y)$.

   (b) Find the flux of the vector field $\mathbf{F}$ of (a) through the sphere of radius 3 centred at the origin in $\mathbb{R}^3$.

8. True or false?

   (a) $\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{r}) = 0$, where $\mathbf{a}$ is a constant vector in $\mathbb{R}^3$, and $\mathbf{r}$ is the vector field $\mathbf{r} = (x, y, z)$.

   (b) $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = 0$ for all scalar fields f on $\mathbb{R}^3$ with continuous second partial derivatives.

   (c) $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) = 0$ for every vector field $\mathbf{F}$ on $\mathbb{R}^3$ with continuous second partial derivatives.

   (d) Suppose $\mathbf{F}$ is a vector field with continuous partial derivatives in the region D, where D is $\mathbb{R}^3$ without the origin. If $\boldsymbol{\nabla} \cdot \mathbf{F} = 0$, then the flux of $\mathbf{F}$ through the sphere of radius 5 with center at the origin is 0.

   (e) Suppose $\mathbf{F}$ is a vector field with continuous partial derivatives in the region D, where D is $\mathbb{R}^3$ without the origin. If $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$ then $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is zero, for every simple and smooth closed curve C in $\mathbb{R}^3$ which avoids the origin.

   (f) If a vector field $\mathbf{F}$ is defined and has continuous partial derivatives everywhere in $\mathbb{R}^3$, and it satisfies $\boldsymbol{\nabla} \cdot \mathbf{F} > 0$, everywhere, then, for every sphere, the flux *out* of one hemisphere is larger than the flux *into* the opposite hemisphere.

   (g) If $\mathbf{r}(t)$ is a path in $\mathbb{R}^3$ with constant curvature κ, then $\mathbf{r}(t)$ parametrizes part of a circle of radius $1/\kappa$.

   (h) The vector field $\mathbf{F} = \left(-\frac{y}{x^2+y^2}\,,\ \frac{x}{x^2+y^2}\,,\ z\right)$ is conservative in its domain, which is $\mathbb{R}^3$, without the z–axis.

   (i) If all flow lines of a vector field in $\mathbb{R}^3$ are parallel to the z–axis, then the circulation of the vector field around every closed curve is 0.

   (j) If the speed of a moving particle is constant, then its acceleration is orthogonal to its velocity.