MATHEMATICS 317 December 2012 Final Exam

1. (a) Consider the parametrized space curve

\[ \mathbf{r}(t) = (\cos(t), \sin(t), t^2) \]

Find a parametric form for the tangent line at the point corresponding to \( t = \pi \).

(b) Find the tangential component \( a_T(t) \) of acceleration, as a function of \( t \), for the parametrized space curve of (a).

2. (a) Let

\[ \mathbf{r}(t) = (2 \sin^3 t, 2 \cos^3 t, 3 \sin t \cos t) \]

Find the unit tangent vector to this parametrized curve at \( t = \pi/3 \), pointing in the direction of increasing \( t \).

(b) Reparametrize the vector function \( \mathbf{r}(t) \) from (a) with respect to arc length measured from the point \( t = 0 \) in the direction of increasing \( t \).

3. (a) Consider the vector field \( \mathbf{F} = (3y, x - 1) \) in \( \mathbb{R}^2 \). Compute the line integral

\[ \int_L \mathbf{F} \cdot d\mathbf{r} \]

where \( L \) is the line segment from \((1,1)\) to \((2,2)\).

(b) Find an oriented path \( C \) from \((2,2)\) to \((1,1)\) such that

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = 4 \]

where \( \mathbf{F} \) is the vector field from (a).

4. (a) Find the curl of the vector field \( \mathbf{F} = (2 + x^2 + z, 0, 3 + x^2 z) \).

(b) Let \( C \) be the curve in \( \mathbb{R}^3 \) from the point \((0,0,0)\) to the point \((2,0,0)\), consisting of three consecutive line segments connecting the points \((0,0,0)\) to \((0,0,3)\), \((0,0,3)\) to \((0,1,0)\), and \((0,1,0)\) to \((2,0,0)\). Evaluate the line integral

\[ \int_C \mathbf{F} \cdot d\mathbf{r} \]

where \( \mathbf{F} \) is the vector field from (a).

5. (a) Consider the surface \( S \) given by the equation

\[ x^2 + z^2 = \cos^2 y \]

Find an equation for the tangent plane to \( S \) at the point \( \left( \frac{1}{2}, \frac{\pi}{4}, \frac{1}{2} \right) \).
(b) Compute the integral
\[ \int \int_S \sin \theta \, dS \]
where \( S \) is the part of the surface from (a) lying between the planes \( y = 0 \) and \( y = \frac{1}{2} \pi \).

6. (a) Let \( S \) be the bucket shaped surface consisting of the cylindrical surface \( y^2 + z^2 = 9 \) between \( x = 0 \) and \( x = 5 \), and the disc inside the \( yz \)-plane of radius 3 centered at the origin. (The bucket \( S \) has a bottom, but no lid.) Orient \( S \) in such a way that the unit normal points outward. Compute the flux of the vector field \( \nabla \times \mathbf{G} \) through \( S \), where \( \mathbf{G} = (x, -z, y) \).

(b) Compute the flux of the vector field \( \mathbf{F} = (2 + z, xz^2, x \cos \theta) \) through \( S \), where \( S \) is as in (a).

7. (a) Find the divergence of the vector field \( \mathbf{F} = (z + \sin \theta, zy, \sin x \cos \theta) \).

(b) Find the flux of the vector field \( \mathbf{F} \) of (a) through the sphere of radius 3 centred at the origin in \( \mathbb{R}^3 \).

8. True or false?

(a) \( \nabla \times (\mathbf{a} \times \mathbf{r}) = 0 \), where \( \mathbf{a} \) is a constant vector in \( \mathbb{R}^3 \), and \( \mathbf{r} \) is the vector field \( \mathbf{r} = (x, y, z) \).

(b) \( \nabla \cdot (\nabla f) = 0 \) for all scalar fields \( f \) on \( \mathbb{R}^3 \) with continuous second partial derivatives.

(c) \( \nabla(\nabla \cdot \mathbf{F}) = 0 \) for every vector field \( \mathbf{F} \) on \( \mathbb{R}^3 \) with continuous second partial derivatives.

(d) Suppose \( \mathbf{F} \) is a vector field with continuous partial derivatives in the region \( D \), where \( D \) is \( \mathbb{R}^3 \) without the origin. If \( \nabla \cdot \mathbf{F} = 0 \), then the flux of \( \mathbf{F} \) through the sphere of radius 5 with center at the origin is 0.

(e) Suppose \( \mathbf{F} \) is a vector field with continuous partial derivatives in the region \( D \), where \( D \) is \( \mathbb{R}^3 \) without the origin. If \( \nabla \times \mathbf{F} = \mathbf{0} \) then \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is zero, for every simple and smooth closed curve \( C \) in \( \mathbb{R}^3 \) which avoids the origin.

(f) If a vector field \( \mathbf{F} \) is defined and has continuous partial derivatives everywhere in \( \mathbb{R}^3 \), and it satisfies \( \nabla \cdot \mathbf{F} > 0 \), everywhere, then, for every sphere, the flux out of one hemisphere is larger than the flux into the opposite hemisphere.

(g) If \( \mathbf{r}(t) \) is a path in \( \mathbb{R}^3 \) with constant curvature \( \kappa \), then \( \mathbf{r}(t) \) parametrizes part of a circle of radius \( 1/\kappa \).

(h) The vector field \( \mathbf{F} = \left( -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, z \right) \) is conservative in its domain, which is \( \mathbb{R}^3 \), without the \( z \)-axis.

(i) If all flow lines of a vector field in \( \mathbb{R}^3 \) are parallel to the \( z \)-axis, then the circulation of the vector field around every closed curve is 0.

(j) If the speed of a moving particle is constant, then its acceleration is orthogonal to its velocity.