MATHEMATICS 317 December 2010 Final Exam

1. Let $\mathbf{r}(t) = \big(3\cos t, 3\sin t, 4t\big)$ be the position vector of a particle as a function of time $t \geq 0$.

   (a) Find the velocity of the particle as a function of time t.

   (b) Find the arclength of its path between $t = 1$ and $t = 2$.

2. Let C be the upper half of the unit circle centred on $(1, 0)$ (i.e. that part of the circle which lies above the x–axis), oriented clockwise. Compute the line integral $\int_C xy\,\mathrm{d}y$.

3. Let S be the surface given by

$$\mathbf{r}(u, v) = \big(u + v\,,\ u^2 + v^2\,,\ u - v\big), \qquad -2 \leq u \leq 2,\ -2 \leq v \leq 2$$

   (a) Find the tangent plane to the surface at the point $(2, 2, 0)$.

   (b) This is a surface you are familiar with. What surface is it (it may be just a portion of one of the following)?

   sphere helicoid ellipsoid saddle parabolic bowl cylinder cone plane

   (c) In which direction does the parametrisation orient the surface? Circle the correct choices: In the (positive / negative) (x / y / z) direction.

4. Let

$$\mathbf{F}(x, y) = \big(y^2 - e^{-y^2} + \sin x\,,\ 2xye^{-y^2} + x\big)$$

   Let C be the boundary of the triangle with vertices $(0, 0)$, $(1, 0)$ and $(1, 2)$, oriented counter–clockwise. Compute

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

5. Let

$$\mathbf{F}(x, y, z) = \left(\frac{y}{x} + x^{1+x^2}\,,\ x^2 - y^{1+y^2}\,,\ \cos^5(\ln z)\right)$$

   (a) Write down the domain D of $\mathbf{F}$.

   (b) Circle the correct statement(s):

      (a) D is connected.

      (b) D is simply connected.

      (c) D is disconnected.

   (c) Compute $\boldsymbol{\nabla} \times \mathbf{F}$.

   (d) Let C be the square with corners $(3 \pm 1, 3 \pm 1)$ in the plane $z = 2$, oriented clockwise (viewed from above, i.e. down z–axis). Compute $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

   (e) Is $\mathbf{F}$ conservative?

6. Let

$$\mathbf{F}(x,y,z) = \left(e^{-y^2} + y^{1+x^2} + \cos(z), -z, y\right)$$

Let S be the surface which consists of two parts:

- the portion of the paraboloid $y^2 + z^2 = 4(x+1)$ satisfying $0 \le x \le 3$ and
- the portion of the sphere $x^2 + y^2 + z^2 = 4$ satisfying $x < 0$

and is oriented outward. Compute

$$\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$$

7. Let

$$\mathbf{F}(x,y,z) = \left(1 + z^{1+z^{1+z}}\,,\ 1 + z^{1+z^{1+z}}\,,\ 1\right)$$

Let S be the portion of the surface

$$x^2 + y^2 = 1 - z^4$$

which is above the xy–plane. What is the flux of $\mathbf{F}$ downward through S?

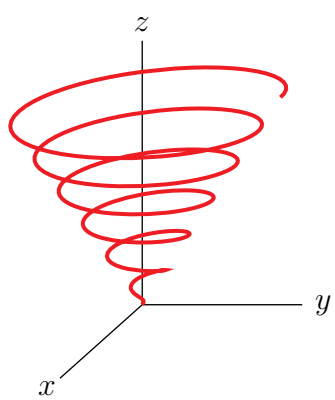
8. Let

$$\mathbf{F}(x,y) = \left(1, yg(y)\right)$$

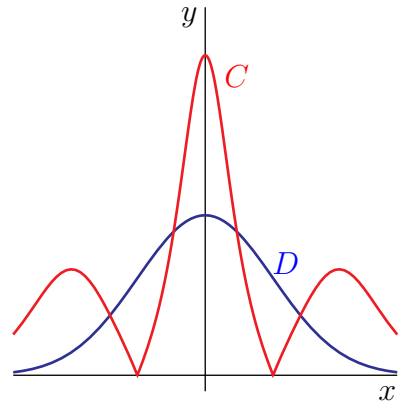
and suppose that $g(y)$ is a function defined everywhere with everywhere continuous partials. Show that for any curve C whose endpoints P and Q lie on the x–axis,

$$\text{distance between } P \text{ and } Q = \left| \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \right|$$

9. (a) In the curve shown below (a helix lying in the surface of a cone), is the curvature increasing, decreasing, or constant as z increases?



(b) Of the two functions shown below, one is a function $f(x)$ and one is its curvature $\kappa(x)$. Which is which?



(c) Let C be the curve of intersection of the cylinder $x^2 + z^2 = 1$ and the saddle $xz = y$. Parametrise C. (Be sure to specify the domain of your parametrisation.)

(d) Let H be the helical ramp (also known as a helicoid) which revolves around the z–axis in a clockwise direction viewed from above, beginning at the y-axis when $z = 0$, and rising 2π units each time it makes a full revolution. Let S be the the portion of H which lies outside the cylinder $x^2 + y^2 = 4$, above the $z = 0$ plane and below the $z = 5$ plane. Choose one of the following functions and give the domain on which the function you have chosen parametrizes S. (Hint: Only one of the following functions is possible.)

(a) $\mathbf{r}(u, v) = \big(u \cos v, u \sin v, u\big)$
(b) $\mathbf{r}(u, v) = \big(u \cos v, u \sin v, v\big)$
(c) $\mathbf{r}(u, v) = \big(u \sin v, u \cos v, u\big)$
(d) $\mathbf{r}(u, v) = \big(u \sin v, u \cos v, v\big)$

(e) Write down a parametrized curve of zero curvature and arclength 1. (Be sure to specify the domain of your parametrisation.)

(f) If $\boldsymbol{\nabla} \cdot \mathbf{F}$ is a constant C on all of $\mathbb{R}^3$, and S is a cube of unit volume such that the flux outward through each side of S is 1, what is C?

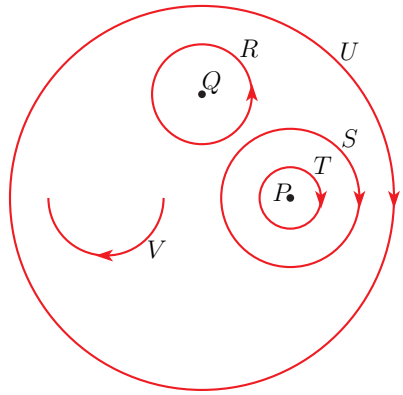
(g) Let

$$\mathbf{F}(x, y) = \big(ax + by\,,\ cx + dy\big)$$

Give the full set of a, b, c and d such that $\mathbf{F}$ is conservative.

(h) If $\mathbf{r}(s)$ has been parametrized by arclength (i.e. s is arclength), what is the arclength of $\mathbf{r}(s)$ between $s = 3$ and $s = 5$?

(i) Let $\mathbf{F}$ be a 2D vector field which is defined everywhere except at the points marked P and Q. Suppose that $\boldsymbol{\nabla} \times \mathbf{F} = 0$ everywhere on the domain of $\mathbf{F}$. Consider the five curves R, S, T, U, and V shown in the picture.



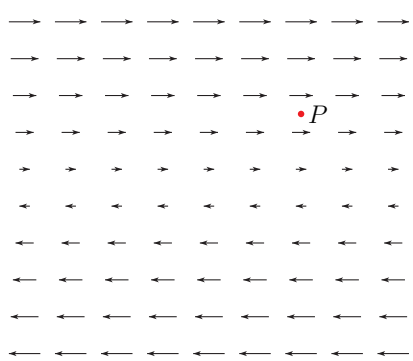
Which of the following is necessarily true?

(1) $\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_T \mathbf{F} \cdot \mathrm{d}\mathbf{r}$
(2) $\int_R \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_T \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_U \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$
(3) $\int_R \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_T \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_U \mathbf{F} \cdot \mathrm{d}\mathbf{r}$
(4) $\int_U \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_R \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{r}$
(5) $\int_V \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$

(j) Write down a 3D vector field $\mathbf{F}$ such that for all closed surfaces S, the volume enclosed by S is equal to

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

(k) Consider the vector field $\mathbf{F}$ in the xy–plane shown below. Is the $\hat{\mathbf{k}}^{\text{th}}$ component of $\boldsymbol{\nabla} \times \mathbf{F}$ at P positive, negative or zero?



10. Say whether the following statements are true or false.

(a) If $\mathbf{F}$ is a 3D vector field defined on all of $\mathbb{R}^3$, and S_1 and S_2 are two surfaces with the same boundary, but $\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \neq \iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$, then $\boldsymbol{\nabla} \cdot \mathbf{F}$ is not zero anywhere.

(b) If $\mathbf{F}$ is a vector field satisfying $\boldsymbol{\nabla} \times \mathbf{F} = 0$ whose domain is not simply-connected, then $\mathbf{F}$ is not conservative.

(c) The osculating circle of a curve C at a point has the same unit tangent vector, unit normal vector, and curvature as C at that point.

(d) A planet orbiting a sun has period proportional to the cube of the major axis of the orbit.

(e) For any 3D vector field $\mathbf{F}$, $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0$.

(f) A field whose divergence is zero everywhere in its domain has closed surfaces S in its domain.

(g) The gravitational force field is conservative.

(h) If $\mathbf{F}$ is a field defined on all of $\mathbb{R}^3$ such that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 3$ for some curve C, then $\boldsymbol{\nabla} \times \mathbf{F}$ is non-zero at some point.

(i) The normal component of acceleration for a curve of constant curvature is constant.

(j) The curve defined by

$$\mathbf{r}_1(t) = \cos(t^4)\,\hat{\boldsymbol{\imath}} + 3t^4\hat{\boldsymbol{\jmath}}, \qquad -\infty < t < \infty,$$

is the same as the curve defined by

$$\mathbf{r}_2(t) = \cos t\,\hat{\boldsymbol{\imath}} + 3t\,\hat{\mathbf{k}}, \qquad -\infty < t < \infty$$