1. Let \( \mathbf{r}(t) = (3 \cos t, 3 \sin t, 4t) \) be the position vector of a particle as a function of time \( t \geq 0 \).

(a) Find the velocity of the particle as a function of time \( t \).
(b) Find the arclength of its path between \( t = 1 \) and \( t = 2 \).

2. Let \( C \) be the upper half of the unit circle centred on \((1, 0)\) (i.e. that part of the circle which lies above the \(x\)-axis), oriented clockwise. Compute the line integral \( \int_C xy \, dy \).

3. Let \( S \) be the surface given by
\[
\mathbf{r}(u, v) = (u + v, u^2 + v^2, u - v), \quad -2 \leq u \leq 2, \quad -2 \leq v \leq 2
\]

(a) Find the tangent plane to the surface at the point \((2, 2, 0)\).
(b) This is a surface you are familiar with. What surface is it (it may be just a portion of one of the following)?

- sphere
- helicoid
- ellipsoid
- saddle
- parabolic bowl
- cylinder
- cone
- plane
(c) In which direction does the parametrisation orient the surface? Circle the correct choices: In the ( positive / negative ) \((x / y / z)\) direction.

4. Let
\[
\mathbf{F}(x, y) = (y^2 - e^{-y^2} \sin x, 2xye^{-y^2} + x)
\]
Let \( C \) be the boundary of the triangle with vertices \((0, 0)\), \((1, 0)\) and \((1, 2)\), oriented counter-clockwise. Compute
\[
\int_C \mathbf{F} \cdot d\mathbf{r}
\]

5. Let
\[
\mathbf{F}(x, y, z) = \left( \frac{y}{x} + x^{1+x^2}, \ x^2 - y^{1+y^2}, \ \cos^5(\ln z) \right)
\]

(a) Write down the domain \( D \) of \( \mathbf{F} \).
(b) Circle the correct statement(s):
   (a) \( D \) is connected.
   (b) \( D \) is simply connected.
   (c) \( D \) is disconnected.
(c) Compute \( \nabla \times \mathbf{F} \).
(d) Let \( C \) be the square with corners \((3 \pm 1, 3 \pm 1)\) in the plane \( z = 2 \), oriented clockwise (viewed from above, i.e. down \(z\)-axis). Compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \).
(e) Is \( \mathbf{F} \) conservative?
6. Let 
\[ \mathbf{F}(x, y, z) = \left( e^{-y^2} + y^{1+x^2} + \cos(z), -z, y \right) \]

Let \( S \) be the surface which consists of two parts:
- the portion of the paraboloid \( y^2 + z^2 = 4(x + 1) \) satisfying \( 0 \leq x \leq 3 \) and
- the portion of the sphere \( x^2 + y^2 + z^2 = 4 \) satisfying \( x < 0 \)

and is oriented outward. Compute
\[
\iint_S \nabla \times \mathbf{F} \cdot \hat{n} \, dS
\]

7. Let 
\[ \mathbf{F}(x, y, z) = \left( 1 + z^{1+z^{1+z}}, 1 + z^{1+z^{1+z}}, 1 \right) \]

Let \( S \) be the portion of the surface
\[ x^2 + y^2 = 1 - z^4 \]

which is above the \( xy \)-plane. What is the flux of \( \mathbf{F} \) downward through \( S \)?

8. Let 
\[ \mathbf{F}(x, y) = \left( 1, yg(y) \right) \]

and suppose that \( g(y) \) is a function defined everywhere with everywhere continuous partials. Show that for any curve \( C \) whose endpoints \( P \) and \( Q \) lie on the \( x \)-axis,
\[
\text{distance between } P \text{ and } Q = \left| \int_C \mathbf{F} \cdot d\mathbf{r} \right|
\]

9. (a) In the curve shown below (a helix lying in the surface of a cone), is the curvature increasing, decreasing, or constant as \( z \) increases?

(b) Of the two functions shown below, one is a function \( f(x) \) and one is its curvature \( \kappa(x) \). Which is which?
(c) Let $C$ be the curve of intersection of the cylinder $x^2 + z^2 = 1$ and the saddle $xz = y$. Parametrise $C$. (Be sure to specify the domain of your parametrisation.)

(d) Let $H$ be the helical ramp (also known as a helicoid) which revolves around the $z$–axis in a clockwise direction viewed from above, beginning at the $y$-axis when $z = 0$, and rising $2\pi$ units each time it makes a full revolution. Let $S$ be the the portion of $H$ which lies outside the cylinder $x^2 + y^2 = 4$, above the $z = 0$ plane and below the $z = 5$ plane. Choose one of the following functions and give the domain on which the function you have chosen parametrizes $S$. (Hint: Only one of the following functions is possible.)

(a) $\mathbf{r}(u, v) = (u \cos v, u \sin v, u)$
(b) $\mathbf{r}(u, v) = (u \cos v, u \sin v, v)$
(c) $\mathbf{r}(u, v) = (u \sin v, u \cos v, u)$
(d) $\mathbf{r}(u, v) = (u \sin v, u \cos v, v)$

(e) Write down a parametrized curve of zero curvature and arclength 1. (Be sure to specify the domain of your parametrisation.)

(f) If $\nabla \cdot \mathbf{F}$ is a constant $C$ on all of $\mathbb{R}^3$, and $S$ is a cube of unit volume such that the flux outward through each side of $S$ is 1, what is $C$?

(g) Let

$$\mathbf{F}(x, y) = (ax + by, cx + dy)$$

Give the full set of $a$, $b$, $c$ and $d$ such that $\mathbf{F}$ is conservative.

(h) If $\mathbf{r}(s)$ has been parametrized by arclength (i.e. $s$ is arclength), what is the arclength of $\mathbf{r}(s)$ between $s = 3$ and $s = 5$?

(i) Let $\mathbf{F}$ be a 2D vector field which is defined everywhere except at the points marked $P$ and $Q$. Suppose that $\nabla \times \mathbf{F} = 0$ everywhere on the domain of $\mathbf{F}$. Consider the five curves $R$, $S$, $T$, $U$, and $V$ shown in the picture.
Which of the following is necessarily true?

(1) \[ \int_S \mathbf{F} \cdot d\mathbf{r} = \int_T \mathbf{F} \cdot d\mathbf{r} \]
(2) \[ \int_R \mathbf{F} \cdot d\mathbf{r} = \int_S \mathbf{F} \cdot d\mathbf{r} = \int_T \mathbf{F} \cdot d\mathbf{r} = \int_U \mathbf{F} \cdot d\mathbf{r} = 0 \]
(3) \[ \int_R \mathbf{F} \cdot d\mathbf{r} + \int_S \mathbf{F} \cdot d\mathbf{r} + \int_T \mathbf{F} \cdot d\mathbf{r} = \int_U \mathbf{F} \cdot d\mathbf{r} \]
(4) \[ \int_U \mathbf{F} \cdot d\mathbf{r} = \int_R \mathbf{F} \cdot d\mathbf{r} + \int_S \mathbf{F} \cdot d\mathbf{r} \]
(5) \[ \int_V \mathbf{F} \cdot d\mathbf{r} = 0 \]

(j) Write down a 3D vector field \( \mathbf{F} \) such that for all closed surfaces \( S \), the volume enclosed by \( S \) is equal to

\[ \iint_S \mathbf{F} \cdot \hat{n} \, dS \]

(k) Consider the vector field \( \mathbf{F} \) in the \( xy \)-plane shown below. Is the \( \hat{k} \)th component of \( \nabla \times \mathbf{F} \) at \( P \) positive, negative or zero?

10. Say whether the following statements are true or false.

(a) If \( \mathbf{F} \) is a 3D vector field defined on all of \( \mathbb{R}^3 \), and \( S_1 \) and \( S_2 \) are two surfaces with the same boundary, but \( \iint_{S_1} \mathbf{F} \cdot \hat{n} \, dS \neq \iint_{S_2} \mathbf{F} \cdot \hat{n} \, dS \), then \( \nabla \cdot \mathbf{F} \) is not zero anywhere.

(b) If \( \mathbf{F} \) is a vector field satisfying \( \nabla \times \mathbf{F} = 0 \) whose domain is not simply-connected, then \( \mathbf{F} \) is not conservative.
(c) The osculating circle of a curve $C$ at a point has the same unit tangent vector, unit normal vector, and curvature as $C$ at that point.

(d) A planet orbiting a sun has period proportional to the cube of the major axis of the orbit.

(e) For any 3D vector field $\mathbf{F}$, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

(f) A field whose divergence is zero everywhere in its domain has closed surfaces $S$ in its domain.

(g) The gravitational force field is conservative.

(h) If $\mathbf{F}$ is a field defined on all of $\mathbb{R}^3$ such that $\int_C \mathbf{F} \cdot d\mathbf{r} = 3$ for some curve $C$, then $\nabla \times \mathbf{F}$ is non-zero at some point.

(i) The normal component of acceleration for a curve of constant curvature is constant.

(j) The curve defined by

$$\mathbf{r}_1(t) = \cos(t^4)\mathbf{i} + 3t^4\mathbf{j}, \quad -\infty < t < \infty,$$

is the same as the curve defined by

$$\mathbf{r}_2(t) = \cos t\mathbf{i} + 3t\mathbf{k}, \quad -\infty < t < \infty.$$