MATHEMATICS 317 December 2005 Final Exam

1. Let $\mathbf{r}(t)$ be a vector valued function. Let $\mathbf{r}'$, $\mathbf{r}''$, and $\mathbf{r}'''$ denote $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$, $\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$ and $\frac{\mathrm{d}^3\mathbf{r}}{\mathrm{d}t^3}$, respectively. Express
$$\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}''\big]$$
in terms of $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{r}''$, and $\mathbf{r}'''$. Select the correct answer.

   (a) $(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''$

   (b) $(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}+(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}'''$

   (c) $(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}'''$

   (d) 0

   (e) None of the above.

2. Say whether the following statements are true (T) or false (F). You may assume that all functions and vector fields are defined everywhere and have derivatives of all orders everywhere.

   (a) The divergence of $\boldsymbol{\nabla}\times\mathbf{F}$ is zero, for every $\mathbf{F}$.

   (b) In a simply connected region, $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ depends only on the endpoints of C.

   (c) If $\boldsymbol{\nabla} f=0$, then f is a constant function.

   (d) If $\boldsymbol{\nabla}\times\mathbf{F}=\mathbf{0}$, then $\mathbf{F}$ is a constant vector field.

   (e) If $\boldsymbol{\nabla}\cdot\mathbf{F}=0$, then $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S=0$ for every closed surface S.

   (f) If $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}=0$ for every closed curve C, then $\boldsymbol{\nabla}\times\mathbf{F}=0$.

   (g) If $\mathbf{r}(t)$ is a path in three space with constant speed $|\mathbf{v}(t)|$, then the acceleration is perpendicular to the tangent vector, i.e. $\mathbf{a}\cdot\hat{\mathbf{T}}=0$.

   (h) If $\mathbf{r}(t)$ is a path in three space with constant curvature κ, then $\mathbf{r}(t)$ parameterizes part of a circle of radius $1/\kappa$.

   (i) Let $\mathbf{F}$ be a vector field and suppose that S_1 and S_2 are oriented surfaces with the same boundary curve C, and C is given the direction that is compatible with the orientations of S_1 and S_2 . Then $\iint_{S1}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S=\iint_{S2}\mathbf{F}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$.

   (j) Let $A(t)$ be the area swept out by the trajectory of a planet from time $t=0$ to time t. The $\frac{\mathrm{d}A}{\mathrm{d}t}$ is constant.

3. Find the speed of a particle with the given position function
$$\mathbf{r}(t)=5\sqrt{2}\,t\,\hat{\boldsymbol{\imath}}+e^{5t}\,\hat{\boldsymbol{\jmath}}-e^{-5t}\,\hat{\mathbf{k}}$$
Select the correct answer:

(a) $|\mathbf{v}(t)| = \left(e^{5t} + e^{-5t}\right)$

(b) $|\mathbf{v}(t)| = \sqrt{10 + 5e^{t} + 5e^{-t}}$

(c) $|\mathbf{v}(t)| = \sqrt{10 + e^{10t} + e^{-10t}}$

(d) $|\mathbf{v}(t)| = 5\left(e^{5t} + e^{-5t}\right)$

(e) $|\mathbf{v}(t)| = 5\left(e^{t} + e^{-t}\right)$

4. Find the correct identity, if f is a function and $\mathbf{G}$ and $\mathbf{F}$ are vector fields. Select the true statement.

   (a) $\boldsymbol{\nabla} \cdot (f\mathbf{F}) = f\boldsymbol{\nabla} \times (\mathbf{F}) + (\boldsymbol{\nabla} f) \times \mathbf{F}$

   (b) $\boldsymbol{\nabla} \cdot (f\mathbf{F}) = f\boldsymbol{\nabla} \cdot (\mathbf{F}) + \mathbf{F} \cdot \boldsymbol{\nabla} f$

   (c) $\boldsymbol{\nabla} \times (f\mathbf{F}) = f\boldsymbol{\nabla} \cdot (\mathbf{F}) + \mathbf{F} \cdot \boldsymbol{\nabla} f$

   (d) None of the above are true.

5. Let S be the part of the paraboloid $z + x^2 + y^2 = 4$ lying between the planes $z = 0$ and $z = 1$. For each of the following, indicate whether or not it correctly parameterizes the surface S.

   (a) $\mathbf{r}(u, v) = u\,\hat{\boldsymbol{\imath}} + v\,\hat{\boldsymbol{\jmath}} + (4 - u^2 - v^2)\,\hat{\mathbf{k}}, \qquad 0 \le u^2 + v^2 \le 1$

   (b) $\mathbf{r}(u, v) = (\sqrt{4 - u}\,\cos v)\,\hat{\boldsymbol{\imath}} + (\sqrt{4 - u}\,\sin v)\,\hat{\boldsymbol{\jmath}} + u\,\hat{\mathbf{k}}, \qquad 0 \le u \le 1,\ 0 \le v \le 2\pi$

   (c) $\mathbf{r}(u, v) = (u\cos v)\,\hat{\boldsymbol{\imath}} + (u\sin v)\,\hat{\boldsymbol{\jmath}} + (4 - u^2)\,\hat{\mathbf{k}}, \qquad \sqrt{3} \le u \le 2,\ 0 \le v \le 2\pi$

6. Let S be the part of the plane

$$x + y + z = 2$$

   that lies in the first octant oriented so that $\hat{\mathbf{n}}$ has a positive $\hat{\mathbf{k}}$ component. Let

$$\mathbf{F} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\mathbf{k}}$$

   Evaluate the flux integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

7. Consider the vector field $\mathbf{F}(x, y, z) = 2x\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}} + 2z\,\hat{\mathbf{k}}$.

   (a) Compute $\boldsymbol{\nabla} \times \mathbf{F}$.

   (b) If C is any path from $(0, 0, 0)$ to (a_1, a_2, a_3) and $\mathbf{a} = a_1\,\hat{\boldsymbol{\imath}} + a_2\,\hat{\boldsymbol{\jmath}} + a_3\,\hat{\mathbf{k}}$, show that $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \mathbf{a} \cdot \mathbf{a}$.

8. Let

$$\mathbf{F} = x\sin y\,\hat{\boldsymbol{\imath}} - y\sin x\,\hat{\boldsymbol{\jmath}} + (x - y)z^2\,\hat{\mathbf{k}}$$

   Use Stokes' theorem to evaluate

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

along the path consisting of the straight line segments successively joining the points $P_0 = (0,0,0)$ to $P_1 = (\pi/2,0,0)$ to $P_2 = (\pi/2,0,1)$ to $P_3 = (0,0,1)$ to $P_4 = (0,\pi/2,1)$ to $P_5 = (0,\pi/2,0)$, and back to $(0,0,0)$.

9. Let S be the hemisphere $x^2+y^2+z^2=1$, $z \geq 0$, oriented with $\hat{\mathbf{n}}$ pointing away from the origin. Evaluate the flux integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

where

$$\mathbf{F} = \big(x + \cos(z^2)\big)\,\hat{\boldsymbol{\imath}} + \big(y + \ln(x^2+z^5)\big)\,\hat{\boldsymbol{\jmath}} + \sqrt{x^2+y^2}\,\hat{\mathbf{k}}$$