MATHEMATICS 317 April 2004 Final Exam

1. At time $t = 0$, NASA launches a rocket which follows a trajectory so that its position at any time t is
$$x = \frac{4\sqrt{2}}{3}t^{3/2},\ y = \frac{4\sqrt{2}}{3}t^{3/2},\ z = t(2-t)$$
   (a) Assuming that the flight ends when $z = 0$, find out how far the rocket travels.
   (b) Find the unit tangent and unit normal to the trajectory at its highest point.
   (c) Also, compute the curvature of the trajectory at its highest point.

2. True or false (reasons must be given):
   (a) If a smooth vector field on $\mathbb{R}^3$ is curl free and divergence free, then its potential is harmonic. By definition, $\phi(x, y, z)$ is harmonic if $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi(x, y, z) = 0$.
   (b) If $\mathbf{F}$ is a smooth conservative vector field on $\mathbb{R}^3$, then its flux through any smooth closed surface is zero.

3. A physicist studies a vector field $\mathbf{F}$ in her lab. She knows from theoretical considerations that $\mathbf{F}$ must be of the form $\mathbf{F} = \nabla \times \mathbf{G}$, for some smooth vector field $\mathbf{G}$. Experiments also show that $\mathbf{F}$ must be of the form
$$\mathbf{F}(x, y, z) = (xz + xy)\hat{\imath} + \alpha(yz - xy)\hat{\jmath} + \beta(yz + xz)\hat{\mathbf{k}}$$
   where α and β are constant.
   (a) Determine α and β.
   (b) Further experiments show that $\mathbf{G} = xyz\hat{\imath} - xyz\hat{\jmath} + g(x, y, z)\hat{\mathbf{k}}$. Find the unknown function $g(x, y, z)$.

4. Recall that if S is a smooth closed surface with outer normal field $\hat{\mathbf{n}}$, then for any smooth function $p(x, y, z)$ on $\mathbb{R}^3$, we have
$$\iint_S p\hat{\mathbf{n}}\ \mathrm{d}s = \iiint_E \nabla p\ \mathrm{d}V$$
   where E is the solid bounded by S. Show that as a consequence, the total force exerted on the surface of a solid body contained in a gas of constant pressure is zero. (Recall that the pressure acts in the direction normal to the surface.)

5. Use Green's theorem to establish that if C is a simple closed curve in the plane, then the area A enclosed by C is given by
$$A = \tfrac{1}{2}\oint_C x\,\mathrm{d}y - y\,\mathrm{d}x$$
   Use this to calculate the area inside the curve $x^{2/3} + y^{2/3} = 1$.

6. Consider the vector field $\mathbf{F}(x, y, z) = -2xy\,\hat{\boldsymbol{\imath}} + \big(y^2 + \sin(xz)\big)\,\hat{\boldsymbol{\jmath}} + (x^2 + y^2)\,\hat{\mathbf{k}}$.

   (a) Calculate $\nabla \cdot \mathbf{F}$.

   (b) Find the flux of $\mathbf{F}$ through the surface S defined by

   $$x^2 + y^2 + (z - 12)^2 = 13^2, \ z \geq 0$$

   using the outward normal to S.

7. Let C be the curve given by the parametric equations:

   $$x = \cos t, \ y = \sqrt{2}\sin t, \ z = \cos t, \ 0 \leq t \leq 2\pi$$

   and let

   $$\mathbf{F} = z\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}} + y^3 z^3\,\hat{\mathbf{k}}$$

   Use Stokes' theorem to evaluate

   $$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$