MATHEMATICS 317 April 2003 Final Exam

1. Find the field line of the vector field $\mathbf{F} = 2y\,\hat{\imath} + \frac{x}{y^2}\,\hat{\jmath} + e^y\hat{\mathbf{k}}$ that passes through $(1, 1, e)$.

2. Let $\mathbf{F} = e^x \sin y\,\hat{\imath} + [ae^x \cos y + bz]\,\hat{\jmath} + cx\,\hat{\mathbf{k}}$. For which values of the constants a, b, c is $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for all closed paths C?

3. Let $\mathbf{F} = \frac{x}{x^2+y^2}\,\hat{\imath} + \frac{y}{x^2+y^2}\,\hat{\jmath} + x^3\,\hat{\mathbf{k}}$. Let P be the path which starts at $(1, 0, 0)$, ends at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}\ln 2\right)$ and follows
$$x^2 + y^2 = 1 \qquad xe^z = 1$$
Find the work done in moving a particle along P in the field $\mathbf{F}$.

4. Let the thin shell S consist of the part of the surface $z^2 = 2xy$ with $x \geq 1$, $y \geq 1$ and $z \leq 2$. Find the mass of S if it has surface density given by $\rho(x, y, z) = 3z$ kg per unit area.

5. Let S be the portion of the paraboloid $x = y^2 + z^2$ that satisfies $x \leq 2y$. Its unit normal vector $\hat{\mathbf{n}}$ is so chosen that $\hat{\mathbf{n}} \cdot \hat{\imath} > 0$. Find the flux of $\mathbf{F} = 2\,\hat{\imath} + z\,\hat{\jmath} + y\,\hat{\mathbf{k}}$ out of S.

6. Let S be the portion of the hyperboloid $x^2 + y^2 - z^2 = 1$ between $z = -1$ and $z = 1$. Find the flux of $\mathbf{F} = (x + e^{yz})\,\hat{\imath} + \big(2yz + \sin(xz)\big)\,\hat{\jmath} + (xy - z - z^2)\,\hat{\mathbf{k}}$ out of S (away from the origin).

7. Evaluate $\displaystyle\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$ where $\mathbf{F} = y\,\hat{\imath} + 2z\,\hat{\jmath} + 3x\,\hat{\mathbf{k}}$ and S is the surface $z = \sqrt{1 - x^2 - y^2}$, $z \geq 0$ and $\hat{\mathbf{n}}$ is a unit normal to S obeying $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} \geq 0$.

8. The following statements may be true or false. Decide which. If true, give a proof. If false, provide a counter–example.

   (a) If f is any smooth function defined in $\mathbb{R}^3$ and if C is any circle, then $\int_C \nabla f \cdot \mathrm{d}\mathbf{r} = 0$.

   (b) There is a vector field $\mathbf{F}$ that obeys $\nabla \times \mathbf{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{\mathbf{k}}$.