MATHEMATICS 317 April 2002 Final Exam

1. Consider a particle travelling in space along the path parametrized by

$$x = \cos^3 t,\ \ y = \sin^3 t,\ \ z = 2\sin^2 t$$

   (a) Calculate the arc length of this path for $0 \le t \le \pi/2$.

   (b) Find the Frenet frame $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$, $\hat{\mathbf{B}}$ for the particle at $t = \pi/6$.

2. Short answers:

   (a) Let S be the level surface $f(x, y, z) = 0$. Why is $\int_C \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} = 0$ for any curve C on S?

   (b) A point moving in space with position $\mathbf{r}(t)$ at time t satisfies the condition $\mathbf{a}(t) = f(t)\mathbf{r}(t)$ for all t for some real valued function f. Why is $\mathbf{v} \times \mathbf{r}$ a constant vector?

   (c) Why is the trajectory of the point in (b) contained in a plane?

   (d) Is the binormal vector, $\hat{\mathbf{B}}$, of a particle moving in space, always orthogonal to the unit tangent vector $\hat{\mathbf{T}}$ and unit normal $\hat{\mathbf{N}}$?

   (e) If the curvature of the path of a particle moving in space is constant, is the acceleration zero when maximum speed occurs?

3. Find the area of the part of the surface $z = y^{3/2}$ that lies above the square $0 \le x, y \le 1$.

4. (a) For which values of the constants α, β and γ is the vector field

$$\mathbf{F}(x, y, z) = \alpha e^y\, \hat{\boldsymbol{\imath}} + (xe^y + \beta \cos z)\, \hat{\boldsymbol{\jmath}} - \gamma y \sin z\, \hat{\mathbf{k}}$$

   conservative?

   (b) For those values of α, β and γ found in part (a), calculate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where C is the curve parametrized by $x = t^2$, $y = e^t$, $z = \pi t$, $0 \le t \le 1$.

5. Use the divergence theorem to find the flux of $x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + 2z\hat{\mathbf{k}}$ through the part of the ellipsoid

$$x^2 + y^2 + 2z^2 = 2$$

   with $z \ge 0$. [Note: the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has volume $\frac{4}{3}\pi abc$.]

6. Use Stokes' theorem to evaluate

$$\oint_C z\,\mathrm{d}x + x\,\mathrm{d}y - y\,\mathrm{d}z$$

   where C is the closed curve which is the intersection of the plane $x + y + z = 1$ with the sphere $x^2 + y^2 + z^2 = 1$. Assume that C is oriented clockwise as viewed from the origin.

7. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = (x^2 - y - 1)\,\hat{\boldsymbol{\imath}} + (e^{\cos y} + z^3)\,\hat{\boldsymbol{\jmath}} + (2xz + z^5)\,\hat{\mathbf{k}}$. Evaluate $\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$ where S is the part of the ellipsoid $x^2 + y^2 + 2z^2 = 1$ with $z \geq 0$.

8. (*Bonus Question*) Let $\mathbf{F}$ be a smooth 3–dimensional vector field such that the flux of $\mathbf{F}$ out of the sphere $x^2 + y^2 + z^2 = a^2$ is equal to $\pi(a^3 + 2a^4)$ for every $a > 0$. Calculate $\boldsymbol{\nabla} \cdot \mathbf{F}(0, 0, 0)$.