# MATHEMATICS 317 December 2001 Final Exam

1. The position of a plane at time $t$ is given by $x = y = \frac{4\sqrt{2}}{3}t^{3/2}$, $z = t(2-t)$ from take–off at $t = 0$ to landing at $t = 2$.

   (a) What is the total distance the plane travels on this flight?

   (b) Find the radius of curvature $\kappa$ at the apex of the flight, which occurs at $t = 1$.

   (c) Two external forces are applied to the plane during the flight: the force of gravity $\mathbf{G} = -Mg\,\hat{\mathbf{k}}$, where $M$ is the mass of the plane and $g$ is a constant; and a friction force $\mathbf{F} = -|\mathbf{v}|^2\mathbf{v}$, where $\mathbf{v}$ is the velocity of the plane. Find the work done by each of these forces during the flight.

   (d) One half–hour later, a bird follows the exact same flight–path as the plane, travelling at a constant speed $v = 3$. One can show that at the apex of the path, i.e. when the bird is at $\left(\frac{4\sqrt{2}}{3}, \frac{4\sqrt{2}}{3}, 1\right)$, the principal unit normal to the path points in the $-\hat{\mathbf{k}}$ direction. Find the bird's (vector) acceleration at that moment.

2. Let $\phi(x, y) = xy$ and let $\mathbf{F} = \nabla\phi$.

   (a) Find an equation for the field line of $\mathbf{F}$ which passes through the point $(3, 2)$. Sketch it and verify that it also passes through $(3, -2)$.

   (b) Find $\int_{(3,-2)}^{(3,2)} \mathbf{F} \cdot d\mathbf{r}$, where the line integral is along the field line of (a).

3. Let $\mathbf{F}(x, y) = (x + 3y)\,\hat{\boldsymbol{\imath}} + (x + y)\,\hat{\boldsymbol{\jmath}}$ and $\mathbf{G}(x, y) = (x + y)\,\hat{\boldsymbol{\imath}} + (2x - 3y)\,\hat{\boldsymbol{\jmath}}$ be vector fields. Find a number $A$ such that for each circle $C$ in the plane

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = A \oint_C \mathbf{G} \cdot d\mathbf{r}$$

4. Let $\mathbf{F}(x, y, z) = -z\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}} + y\,\hat{\mathbf{k}}$ be a vector field. Use Stokes' theorem to evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $C$ is the intersection of the plane $z = y$ and the ellipsoid $\frac{x^2}{4} + \frac{y^2}{2} + \frac{z^2}{2} = 1$.

5. Let $S$ denote the portion of the paraboloid $z = 1 - \frac{1}{4}x^2 - y^2$ for which $z \geq 0$. Orient $S$ so that its unit normal has a positive $\hat{k}$ component. Let

$$\mathbf{F}(x, y, z) = (3y^2 + z)\,\hat{\boldsymbol{\imath}} + (x - x^2)\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$$

   Evaluate the surface integral $\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$.

6. Let $\mathbf{F}(x, y) = \frac{y^3}{(x^2+y^2)^2}\hat{\boldsymbol{\imath}} - \frac{xy^2}{(x^2+y^2)^2}\hat{\boldsymbol{\jmath}}$, $(x, y) \neq (0, 0)$.

   (a) Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $C$ is the unit circle in the $xy$–plane, positively oriented.

   (b) Use (a) and Green's theorem to find $\oint_{C_0} \mathbf{F} \cdot d\mathbf{r}$ where $C_0$ is the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$, positively oriented.