MATHEMATICS 317 December 2000 Final Exam

1. A skier descends the hill $z = \sqrt{4 - x^2 - y^2}$ along a trail with parameterization

$$x = \sin(2\theta), \qquad y = 1 - \cos(2\theta), \qquad z = 2\cos\theta, \qquad 0 \le \theta \le \frac{\pi}{2}$$

Let P denote the point on the trail where $x = 1$.

(a) Find Frenet frame $\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}$ and the curvature κ of the ski trail at the point P.

(b) The skier's acceleration at P is $\mathbf{a} = (-2, 3, -2\sqrt{2})$. Find, at P,

(i) the rate of change of the skier's speed and

(ii) the skier's velocity (a vector).

2. Consider the following force field, in which m, n, p, q are constants:

$$\mathbf{F} = (mxyz + z^2 - ny^2)\,\hat{\boldsymbol{\imath}} + (x^2z - 4xy)\,\hat{\boldsymbol{\jmath}} + (x^2y + pxz + qz^3)\,\hat{\mathbf{k}}$$

(a) Find all values of m, n, p, q such that $\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$ for all piecewise smooth closed curves $\mathcal{C}$ in $\mathbb{R}^3$.

(b) For every possible choice of m, n, p, q in (a), find the work done by $\mathbf{F}$ in moving a particle from the bottom to the top of the sphere $x^2 + y^2 + z^2 = 2z$. (The direction of $\hat{\mathbf{k}}$ defines "up".)

3. Let a and b be positive constants, and let $\mathcal{S}$ be the part of the conical surface

$$a^2z^2 = b^2(x^2 + y^2)$$

where $0 \le z \le b$. Consider the surface integral

$$I = \iint_{\mathcal{S}} (x^2 + y^2)\,\mathrm{d}S.$$

(a) Express I as a double integral over a disk in the xy-plane.

(b) Use the parametrization $x = t\cos\theta$, $y = t\sin\theta$, etc., to express I as a double integral over a suitable region in the $t\theta$–plane.

(c) Evaluate I using the method of your choice.

4. Let $\mathcal{S}$ be the curved surface below, oriented by the outward normal:

$$x^2 + y^2 + 2(z - 1)^2 = 6, \qquad z \ge 0.$$

(E.g., at the high point of the surface, the unit normal is $\hat{\mathbf{k}}$.) Define

$$\mathbf{G} = \nabla \times \mathbf{F}, \qquad \text{where} \qquad \mathbf{F} = (xz - y^3\cos z)\,\hat{\boldsymbol{\imath}} + x^3e^z\,\hat{\boldsymbol{\jmath}} + xyze^{x^2+y^2+z^2}\,\hat{\mathbf{k}}.$$

Find $\iint_{\mathcal{S}} \mathbf{G} \cdot \hat{\mathbf{n}}\,\mathrm{d}S$.

5. Let $\mathcal{R}$ be the part of the solid cylinder $x^2 + (y-1)^2 \le 1$ satisfying $0 \le z \le y^2$; let $\mathcal{S}$ be the boundary of $\mathcal{R}$. Given $\mathbf{F} = x^2\,\hat{\imath} + 2y\,\hat{\jmath} - 2z\,\hat{\mathbf{k}}$,

   (a) Find the total flux of $\mathbf{F}$ outward through $\mathcal{S}$.

   (b) Find the total flux of $\mathbf{F}$ outward through the (vertical) cylindrical sides of $\mathcal{S}$.
   Hint: $\displaystyle\int_0^\pi \sin^n\theta\,\mathrm{d}\theta = \frac{n-1}{n}\int_0^\pi \sin^{n-2}\theta\,\mathrm{d}\theta$ for $n = 2, 3, 4, \ldots$.

6. Three quickies:

   (a) A moving particle has velocity and acceleration vectors that satisfy $|\mathbf{v}| = 1$ and $|\mathbf{a}| = 1$ at all times. Prove that the curvature of this particle's path is a constant; evaluate this constant.

   (b) A moving particle with position $\mathbf{r}(t) = (x(t), y(t), z(t))$ satisfies

   $$\mathbf{a} = f(\mathbf{r}, \mathbf{v})\mathbf{r}$$

   for some scalar–valued function f. Prove that $\mathbf{r} \times \mathbf{v}$ is constant.

   (c) Calculate $\iint_{\mathcal{S}} (x\,\hat{\imath} - y\,\hat{\jmath} + z^2\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\mathrm{d}S$, where $\mathcal{S}$ is the boundary of any solid right circular cylinder of radius b with one base in the plane $z = 1$ and the other base in the plane $z = 3$.

7. Let $u = u(x, y, z)$ be a solution of Laplace's Equation,

   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

   in $\mathbb{R}^3$. Let $\mathcal{R}$ be a smooth solid in $\mathbb{R}^3$.

   (a) Prove that the total flux of $\mathbf{F} = \nabla u$ out through the boundary of $\mathcal{R}$ is zero.

   (b) Prove that the total flux of $\mathbf{G} = u\nabla u$ out through the boundary of $\mathcal{R}$ equals

   $$\iiint_{\mathcal{R}} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] \mathrm{d}V$$