Special Instructions:
No memory aids, calculators or communication devices are allowed.

Rules Governing Formal Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBC card for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   i. speaking or communicating with other examination candidates, unless otherwise authorized;
   ii. purposely exposing written papers to the view of other examination candidates or imaging devices;
   iii. purposely viewing the written papers of other examination candidates;
   iv. using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   v. using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) — (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).
1. (a) Find the curvature of $y = e^x$ at $(0, 1)$.

(b) Find the equation of the circle best fitting $y = e^x$ at $(0, 1)$.
2. The surface $z = x^2 + y^2$ is sliced by the plane $x = y$. The resulting curve is oriented from $(0, 0, 0)$ to $(1, 1, 2)$.

(a) Sketch the curve from $(0, 0, 0)$ to $(1, 1, 2)$.

(b) Sketch $\hat{T}$, $\hat{N}$ and $\hat{B}$ at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

(c) Find the torsion at $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. 

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3. (a) For which value(s) of the constants $a, b$ is the vector field

$$\mathbf{F} = (2x \sin(\pi y) - e^z)i + (ax^2 \cos(\pi y) - 3e^z)j - (x + by)e^z\mathbf{k}$$

conservative?

(b) Let $\mathbf{F}$ be a conservative field from part (a). Find all functions $\phi$ for which $\mathbf{F} = \nabla \phi$.

(c) Let $\mathbf{F}$ be a conservative field from part (a). Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $C$ is the intersection of $y = x$ and $z = \ln(1 + x)$ from $(0, 0, 0)$ to $(1, 1, \ln 2)$.

(d) Evaluate $\int_C \mathbf{G} \cdot d\mathbf{r}$ where

$$\mathbf{G} = (2x \sin(\pi y) - e^z) \mathbf{i} + (\pi x^2 \cos(\pi y) - 3e^z) \mathbf{j} - xe^z \mathbf{k}$$

and $C$ is the intersection of $y = x$ and $z = \ln(1 + x)$ from $(0, 0, 0)$ to $(1, 1, \ln 2)$.
4. Let the thin shell $S$ consist of the part of the surface $z^2 = 2xy$ with $x \geq 1$, $y \geq 1$ and $0 \leq z \leq 2$. Find the mass of $S$ if it has surface density given by $\rho(x, y, z) = 3z$ kg per unit area.

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5. Let $F = (x^2 + y^2 + z^2)\hat{i} + (e^{x^2 + y^2})\hat{j} + (3 + x + z)\hat{k}$ and let $S$ be the part of the surface $x^2 + y^2 + z^2 = 2az + 3a^2$ having $z \geq 0$, oriented with normal pointing away from the origin. Here $a > 0$ is a constant. Compute the flux of $F$ through $S$. 

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6. Let $C$ be the counterclockwise boundary of the rectangle with vertices $(1,0)$, $(3,0)$, $(3,1)$ and $(1,1)$. Evaluate

$$\oint_C (3y^2 + 2xe^{y^2}) \, dx + (2yx^2e^{y^2}) \, dy$$

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7. Let $S$ be the part of the half cone

$$z = \sqrt{x^2 + y^2}, \quad y \geq 0,$$

that lies below the plane $z = 1$.

(a) Find a parametrization for $S$.

(b) Calculate the flux of the velocity field

$$\mathbf{v} = x\hat{i} + y\hat{j} - 2z\hat{k}$$

downward through $S$.

(c) A vector field $\mathbf{F}$ has curl $\nabla \times \mathbf{F} = x\hat{i} + y\hat{j} - 2z\hat{k}$. On the $xz$-plane, the vector field $\mathbf{F}$ is constant with $\mathbf{F}(x, 0, z) = \hat{j}$. Given this information, calculate

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where $C$ is the half circle

$$x^2 + y^2 = 1, \quad z = 1, \quad y \geq 0$$

oriented from $(-1, 0, 1)$ to $(1, 0, 1)$. 

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8. A region $R$ is bounded by a simple closed curve $C$. The curve $C$ is oriented such that $R$ lies to the left of $C$ when walking along $C$ in the direction of $C$. Determine whether or not each of the following expressions is equal to the area of $R$. You must justify your conclusions.

(a) $\frac{1}{2} \int_C -y \, dx + x \, dy$

(b) $\frac{1}{2} \int_C -x \, dx + y \, dy$

(c) $\int_C y \, dx$

(d) $\int_C 3y \, dx + 4x \, dy$
9. Say whether each of the following statements is true or false and explain why.

(a) A moving particle has velocity and acceleration vectors that satisfy $|v| = 1$ and $|a| = 1$ at all times. Then the curvature of this particle’s path is a constant.

(b) If $\mathbf{F}$ is any smooth vector field defined in $\mathbb{R}^3$ and if $S$ is any sphere, then

$$\iint_S \nabla \times \mathbf{F} \cdot \hat{n} \, dS = 0$$

Here $\hat{n}$ is the outward normal to $S$.

(c) If $\mathbf{F}$ and $\mathbf{G}$ are smooth vector fields in $\mathbb{R}^3$ and if $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{G} \cdot d\mathbf{r}$ for every circle $C$, then $\mathbf{F} = \mathbf{G}$.
Curves

- \( \mathbf{a}(t) = \frac{d^2\mathbf{a}}{dt^2}(t) \mathbf{T}(t) + \kappa(t) \left( \frac{d\mathbf{a}}{dt}(t) \right)^2 \mathbf{N}(t) \)
- \( \kappa(t) = \frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{v}(t)\|^3} \) \( \kappa(t) = \frac{\left| \frac{dx}{dt}(t) \frac{d^2y}{dt^2}(t) - \frac{dy}{dt}(t) \frac{d^2z}{dt^2}(t) \right|}{\left[ \left( \frac{dx}{dt}(t) \right)^2 + \left( \frac{dy}{dt}(t) \right)^2 \right]^{3/2}} \) \( \kappa(x) = \frac{\left| \frac{d^2y}{dx^2}(x) \right|}{\left[ 1 + \left( \frac{dy}{dx}(x) \right)^2 \right]^{3/2}} \)
- \( \tau(t) = \frac{\left( \mathbf{v}(t) \times \mathbf{a}(t) \right) \cdot \frac{d\mathbf{a}}{dt}(t)}{\|\mathbf{v}(t) \times \mathbf{a}(t)\|^2} \)
- \( \mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} \mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t) = \frac{\frac{d\mathbf{T}}{dt}(t)}{\|\frac{d\mathbf{T}}{dt}(t)\|} \mathbf{B}(t) = \mathbf{\hat{T}}(t) \times \mathbf{\hat{N}}(t) = \frac{\mathbf{v}(t) \times \mathbf{a}(t)}{\|\mathbf{v}(t) \times \mathbf{a}(t)\|} \)
- \( \frac{dT}{ds}(s) = \kappa(s) \mathbf{\hat{N}}(s) \)
- \( \frac{d\mathbf{N}}{ds}(s) = \tau(s) \mathbf{B}(s) - \kappa(s) \mathbf{\hat{T}}(s) \)
- \( \frac{d\mathbf{B}}{ds}(s) = -\tau(s) \mathbf{\hat{N}}(s) \)
- the centre of curvature for \( \mathbf{r}(t) \) is \( \mathbf{r}(t) + \frac{1}{\kappa(t)} \mathbf{\hat{N}}(t) \)

Surface Integrals

- \( \text{Parametrized Surfaces. If the surface is parametrized by} \mathbf{r}(u, v), \text{then} \)
  \[ dS = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \, du \, dv \quad \mathbf{n}dS = \pm \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \, du \, dv \]
- \( \text{Level Surfaces. If the surface is} G(x, y, z) = 0, \text{then} \)
  \[ dS = \left| \nabla G \right| \, dx \, dy = \left| \nabla G \cdot \mathbf{i} \right| \, dy \, dz = \left| \nabla G \cdot \mathbf{j} \right| \, dx \, dz \]
  \[ \mathbf{n}dS = \pm \frac{\nabla G}{\nabla G \cdot \mathbf{k}} \, dx \, dy = \pm \frac{\nabla G}{\nabla G \cdot \mathbf{j}} \, dy \, dz = \pm \frac{\nabla G}{\nabla G \cdot \mathbf{i}} \, dx \, dz \]
- \( \text{Graphs. If the surface is} z = f(x, y) \text{ or } x = g(y, z) \text{ or } y = h(x, z) \text{ then} \)
  \[ dS = \sqrt{1 + \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2} \, dx \, dy \]
  \[ \mathbf{n}dS = \pm \left[ -\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k} \right] \, dx \, dy \]
  \[ = \sqrt{1 + \left( \frac{\partial g}{\partial y} \right)^2 + \left( \frac{\partial g}{\partial z} \right)^2} \, dy \, dz \]
  \[ = \pm \left[ \mathbf{i} - \frac{\partial g}{\partial y} \mathbf{j} - \frac{\partial g}{\partial z} \mathbf{k} \right] \, dy \, dz \]
- \( \text{Coordinate Systems} \)
  - \( \text{Cylindrical coordinates} \)
    \[ x = r \cos \theta \quad y = r \sin \theta \quad z = z \quad dV = r \, dr \, d\theta \, dz \]
  - \( \text{Spherical coordinates} \)
    \[ x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi \quad dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi \]
Vector Identities

Gradient
1. $\nabla (f + g) = \nabla f + \nabla g$
2. $\nabla (cf) = c \nabla f$, for any constant $c$
3. $\nabla (fg) = f \nabla g + g \nabla f$
4. $\nabla (f/g) = (g \nabla f - f \nabla g)/g^2$ at points $x$ where $g(x) \neq 0$.
5. $\nabla (F \cdot G) = F \times (\nabla \times G) - (\nabla \times F) \times G + (G \cdot \nabla) F + (F \cdot \nabla) G$

Divergence
6. $\nabla \cdot (F + G) = \nabla \cdot F + \nabla \cdot G$
7. $\nabla \cdot (cF) = c \nabla \cdot F$, for any constant $c$
8. $\nabla \cdot (fF) = f \nabla \cdot F + F \cdot \nabla f$
9. $\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$

Curl
10. $\nabla \times (F + G) = \nabla \times F + \nabla \times G$
11. $\nabla \times (cF) = c \nabla \times F$, for any constant $c$
12. $\nabla \times (fF) = f \nabla \times F + \nabla f \times F$
13. $\nabla \times (F \times G) = F(\nabla \cdot G) - (\nabla \cdot F)G + (G \cdot \nabla) F - (F \cdot \nabla) G$

Laplacian
14. $\nabla^2 (f + g) = \nabla^2 f + \nabla^2 g$
15. $\nabla^2 (cf) = c \nabla^2 f$, for any constant $c$
16. $\nabla^2 (fg) = f \nabla^2 g + 2 \nabla f \cdot \nabla g + g \nabla^2 f$

Degree Two
17. $\nabla \cdot (\nabla \times F) = 0$
18. $\nabla \times (\nabla f) = 0$
19. $\nabla \cdot \left[ f(\nabla g \times \nabla h) \right] = \nabla f \cdot (\nabla g \times \nabla h)$
20. $\nabla \cdot (f \nabla g - g \nabla f) = f \nabla^2 g - g \nabla^2 f$
21. $\nabla \times (\nabla \times F) = \nabla (\nabla \cdot F) - \nabla^2 F$
Integral Identities for Vector Fields

The Divergence Theorem and Variations

\[ \iiint_V \nabla \cdot \mathbf{F} \, dV = \iint_{\partial V} \mathbf{F} \cdot \mathbf{n} \, dS \]

\[ \iiint_V \nabla f \, dV = \iint_{\partial V} f \mathbf{n} \, dS \]

\[ \iiint_V \nabla \times \mathbf{F} \, dV = \iint_{\partial V} \mathbf{n} \times \mathbf{F} \, dS \]

Stokes’ Theorem

\[ \iiint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} \]

Green’s theorem

\[ \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dxdy = \iint_R \left[ \frac{\partial P_2}{\partial x} - \frac{\partial P_1}{\partial y} \right] \, dxdy = \oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} \]