

Discrete Probability Distributions

Uniform Distribution

Experiment obeys: all outcomes equally probable

Random variable: outcome

Probability distribution: if k is the number of possible outcomes,

$$p(x) = \begin{cases} \frac{1}{k} & \text{if } x \text{ is a possible outcome} \\ 0 & \text{otherwise} \end{cases}$$

Example: tossing a fair die ($k = 6$)

Bernoulli Distribution

Experiment obeys: (1) a single trial with two possible outcomes (success and failure)

$$(2) P(\{\text{trial is successful}\}) = p$$

Random variable: number of successful trials (zero or one)

Probability distribution: $p(x) = p^x(1-p)^{n-x}$

Mean and variance: $\mu = p$, $\sigma^2 = p(1-p)$

Example: tossing a fair coin once

Binomial Distribution

Experiment obeys: (1) n repeated trials

(2) each trial has two possible outcomes (success and failure)

$$(3) P(\{i^{\text{th}} \text{ trial is successful}\}) = p \text{ for all } i$$

(4) the trials are independent

Random variable: number of successful trials

Probability distribution: $b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$

Mean and variance: $\mu = np$, $\sigma^2 = np(1-p)$

Example: tossing a fair coin n times

Approximations: (1) $b(x; n, p) \approx p(x; \lambda = pn)$ if $p \ll 1$, $x \ll n$ (Poisson approximation)

(2) $b(x; n, p) \approx n(x; \mu = pn, \sigma = \sqrt{np(1-p)})$ if $np \gg 1$, $n(1-p) \gg 1$
(Normal approximation)

Geometric Distribution

Experiment obeys: (1) indeterminate number of repeated trials

(2) each trial has two possible outcomes (success and failure)

$$(3) P(\{i^{\text{th}} \text{ trial is successful}\}) = p \text{ for all } i$$

(4) the trials are independent

Random variable: trial number of first successful trial

Probability distribution: $p(x) = p(1-p)^{x-1}$

Mean and variance: $\mu = \frac{1}{p}$, $\sigma^2 = \frac{1-p}{p^2}$

Example: repeated attempts to start an engine, or playing a lottery until you win

Negative Binomial Distribution

Experiment obeys: (1) indeterminate number of repeated trials

(2) each trial has two possible outcomes (success and failure)

$$(3) P(\{i^{\text{th}} \text{ trial is successful}\}) = p \text{ for all } i$$

(4) the trials are independent

(5) keep going until r^{th} success

Random variable: trial number on which r^{th} success occurs

Probability distribution: $b^*(x; r, p) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$

Mean and variance: $\mu = \frac{r}{p}$, $\sigma^2 = \frac{r(1-p)}{p^2}$

Example: fabricating r nondefective computer chips

Poisson Distribution

Experiment obeys: count the number of occurrences of some event in a specified time interval or in a specified region of space where:

- (1) the events occur at a point in time or space
- (2) the number of events occurring in one region is independent of the number occurring in any disjoint region
- (3) the probability of more than one event occurring at the same point is negligible
- (4) the probability of n events in region #1 is the same as the probability of n events in region #2, when the regions have the same size

Random variable: number of events occurring in the given time interval or region of space

Probability distribution: $p(x; \lambda) = \frac{e^{-\lambda} \lambda^x}{x!}$ where λ is the average number of events in the given region

Mean and variance: $\mu = \lambda, \sigma^2 = \lambda$

Example: telephone calls arriving at a switchboard in a specified one hour period

Hypergeometric Distribution

Experiment obeys: (1) a random sample of size n is selected from N items

(2) there are k items of one type (called successes) and $N - k$ items of another type (called failures)

Random variable: number of successes selected

Probability distribution: $h(x; N, n, k) = \frac{\binom{k}{x} \binom{N-k}{n-x}}{\binom{N}{n}}$

Mean and variance: $\mu = n \frac{k}{N}, \sigma^2 = \frac{N-n}{N-1} n \frac{k}{N} \left(1 - \frac{k}{N}\right)$

Example: selecting a random sample of 5 spark plugs from a batch of 40 of which 3 are defective