Partial Fractions Examples

Partial fractions is the name given to a technique, used, for example, in evaluating integrals that is used to express any ratio of polynomials in a simple standard form. A ratio of polynomials is called a rational function. Suppose that $N(z)$ and $D(z)$ are polynomials. Partial fractions expresses $\frac{N(z)}{D(z)}$ as the sum of a polynomial $P(z)$ (this term is present if only if the degree of $N(z)$ is bigger than or equal to the degree of $D(z)$) and a number of rational functions of the particularly simple forms $\frac{A}{(az+b)^n}$.

Example 1: $\frac{N(z)}{D(z)} = \frac{z^3 + 6z^2 + 17z + 16}{z^2 + 4z + 3}$

Step 1. The first step is applied only when the degree of the numerator $N(z)$ is at least as large as the degree of the denominator $D(z)$. In this example, the numerator has degree three and the denominator has degree two. As $3 \geq 2$, we must execute the first step, which is to write $\frac{N(z)}{D(z)}$ in the form

$$\frac{N(z)}{D(z)} = P(z) + \frac{R(z)}{D(z)}$$

with $P(z)$ being a polynomial and $R(z)$ being a polynomial of degree strictly smaller than the degree, 2, of $D(z)$. This step is accomplished by long division – the same long division you learned in public school with the base 10 replaced by $z$. We start by observing that to get from the highest degree term in the denominator ($z^2$) to the highest degree term in the numerator ($z^3$), we have to multiply by $z$. So we write,

$${\begin{array}{c|ccccc} z & z^3 + 6z^2 + 17z + 16 \\ \hline z^2 + 4z + 3 & z \\ \end{array}}$$

Now we subtract $z$ times the denominator $z^2 + 4z + 3$, which is $z^3 + 4z^2 + 3z$, from the numerator.

$${\begin{array}{c|ccccc} z & z^3 + 6z^2 + 17z + 16 \\ \hline z^2 + 4z + 3 & z^3 + 6z^2 + 17z + 16 \\ \hline 2z^2 + 14z + 16 & 2z^2 + 14z + 16 \\ \end{array}}$$

The remainder was $2z^2 + 14z + 16$. To get from the highest degree term in the denominator ($z^2$) to the highest degree term in the remainder ($2z^2$), we have to multiply by 2. So we write,
\[ z^2 + 4z + 3 \div \frac{z + 2}{z^3 + 6z^2 + 17z + 16} = \frac{z^3 + 4z^2 + 3z}{2z^2 + 14z + 16} \]

Now we subtract 2 times the denominator \( z^2 + 4z + 3 \), which is \( 2z^2 + 8z + 6 \), from the remainder.

\[ z^2 + 4z + 3 \div \frac{z + 2}{z^3 + 6z^2 + 17z + 16} = \frac{z(z^2 + 4z + 3)}{2z^2 + 14z + 16} \]

This leaves a remainder of \( 6z + 10 \). Because the remainder has degree 1, which is smaller than the degree of the denominator, which is 2, we stop.

In this example, when we subtracted \( z(z^2 + 4z + 3) \) and \( 2(z^2 + 4z + 3) \) from \( z^3 + 6z^2 + 17z + 16 \) we ended up with \( 6z + 10 \). That is,

\[ z^3 + 6z^2 + 17z + 16 - z(z^2 + 4z + 3) - 2(z^2 + 4z + 3) = 6z + 10 \]

or, collecting the two terms proportional to \( (z^2 + 4z + 3) \)

\[ z^3 + 6z^2 + 17z + 16 - (z + 2)(z^2 + 4z + 3) = 6z + 10 \]

Moving the \((z + 2)(z^2 + 4z + 3)\) to the right hand side and dividing the whole equation by \( z^2 + 4z + 3 \) gives

\[ \frac{z^3 + 6z^2 + 17z + 16}{z^2 + 4z + 3} = z + 2 + \frac{6z + 10}{z^2 + 4z + 3} \]

This is of the form \( \frac{N(z)}{D(z)} = P(z) + \frac{R(z)}{D(z)} \), with the degree of \( R(z) \) strictly smaller than the degree of \( D(z) \), which is what we wanted. Observe that \( R(z) \) is the final remainder of the long division procedure and \( P(z) \) is at the top of the long division computation.
**Step 2.** The second step is to factor the denominator \( D(z) = z^2 + 4z + 3 \). As this is of degree two we can use the high school formula \( \frac{-b\pm\sqrt{b^2-4ac}}{2a} \) to find the roots. Instead, I will pretend that we do not know such a formula and show you a trick that often works for higher degree polynomials too. The trick exploits the fact that most polynomials that appear in homework assignments and on tests have integer coefficients and some integer roots. Any integer root of a polynomial that has integer coefficients, like \( D(z) = z^2 + 4z + 3 \), must divide the constant term of the polynomial exactly. So any integer root of \( z^2 + 4z + 3 \) must divide 3 exactly. The only integers which can be roots of \( D(z) \) are \( \pm 1 \) and \( \pm 3 \). To test if \( +1 \) is a root, we sub it into \( D(z) \):

\[
D(1) = (1)^2 + 4(1) + 3 = 8
\]

As \( D(1) \neq 0 \), \( 1 \) is not a root of \( D(z) \). To test if \( -1 \) is a root, we sub it into \( D(z) \):

\[
D(-1) = (-1)^2 + 4(-1) + 3 = 1 - 4 + 3 = 0
\]

As \( D(-1) = 0 \), \( -1 \) is a root of \( D(z) \). As \( -1 \) is a root of \( D(z) \), \( (z - (-1)) = (z + 1) \) must divide \( D(z) \) exactly. We can factor the \( (z + 1) \) out of \( D(z) = z^2 + 4z + 3 \) by long division once again.

\[
\begin{array}{c|cc}
& z^2 + 4z + 3 \\
\hline
z + 1 & \multicolumn{2}{c}{z(z + 1)} \\
& z + 1 & \multicolumn{1}{c}{3(z + 1)} \\
\hline
& z^2 + z & 3z + 3 \\
& & 3z + 3 \\
& & 0
\end{array}
\]

This time, when we subtracted \( z(z + 1) \) and \( 3(z + 1) \) from \( z^2 + 4z + 3 \) we ended up with 0. Hence

\[
z^2 + 4z + 3 - z(z + 1) - 3(z + 1) = 0
\]

or

\[
z^2 + 4z + 3 = (z + 3)(z + 1)
\]

We have reached the end of step 2. At this point we have

\[
\frac{z^3 + 6z^2 + 17z + 16}{z^2 + 4z + 3} = z + 2 + \frac{6z + 10}{(z + 1)(z + 3)}
\]

**Step 3.** The third step is to write \( \frac{6z + 10}{(z + 1)(z + 3)} \) in the form

\[
\frac{6z + 10}{(z + 1)(z + 3)} = \frac{A}{z + 1} + \frac{B}{z + 3}
\]
for some constants $A$ and $B$. To determine the values of the constants $A$, $B$, we put the right hand side back over the common denominator $(z + 1)(z + 3)$.

\[
\frac{6z + 10}{(z + 1)(z + 3)} = \frac{A}{z + 1} + \frac{B}{z + 3} = \frac{A(z + 3) + B(z + 1)}{(z + 1)(z + 3)}
\]

The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.

\[
6z + 10 = A(z + 3) + B(z + 1)
\]

There are a couple of different ways to determine the values of $A$ and $B$ from this equation. The conceptually clearest procedure is to write the right hand side as a polynomial in standard form (i.e. collect up all $z$ terms and all constant terms)

\[
6z + 10 = (A + B)z + (3A + B)
\]

For these two polynomials to be the same, the coefficient of $z$ on the left hand side and the coefficient of $z$ on the right hand side must be the same. Similarly the coefficients of $z^0$ must match. This gives us a system of two equations

\[
\begin{align*}
A + B &= 6 \\
3A + B &= 10
\end{align*}
\]

in the two unknowns $A, B$. We can solve this system by using the first equation, $A + B = 6$, to determine $A$ in terms of $B$, $A = 6 - B$. Substituting this into the remaining equation eliminates the $A$’s from this equation, leaving one equation in the one unknown $B$.

\[
\begin{align*}
A &= 6 - B \\
3A + B &= 10
\end{align*}
\]

\[
\Rightarrow \quad 3(6 - B) + B = 10
\]

\[
\Rightarrow \quad -2B = -8
\]

So $B = 4$ and $A = 6 - B = 2$.

There is a second, more efficient, procedure for determining $A$ and $B$ from

\[
6z + 10 = A(z + 3) + B(z + 1)
\]

This equation must be true for all values of $z$. In particular, it must be true for $z = -1$. When $z = -1$, the factor $(z + 1)$ multiplying $B$ is exactly zero. So $B$ disappears from the equation, leaving us with an easy equation to solve for $A$:

\[
6z + 10 \bigg|_{z=-1} = A(z + 3) \bigg|_{z=-1} + B(z + 1) \bigg|_{z=-1} \Rightarrow 4 = 2A \implies A = 2
\]
Similarly, this equation must also be true for \( z = -3 \). When \( z = -3 \), the factor \((z + 3)\) multiplying \( A \) is exactly zero. So \( A \) disappears from the equation, leaving us with an easy equation to solve for \( B \):

\[
6z + 10 \bigg|_{z=-3} = A(z + 3) \bigg|_{z=-3} + B(z + 1) \bigg|_{z=-3} \implies -8 = -2B \implies B = 4
\]

Subbing these values of \( A \) and \( B \) into \( 6z + 10 \) we have

\[
\frac{z^3 + 6z^2 + 17z + 16}{z^2 + 4z + 3} = z + 2 + \frac{2}{z + 1} + \frac{4}{z + 3}
\]

**Example 2:**

\[
\frac{N(z)}{D(z)} = \frac{z^4 + 9z^3 + 31z^2 + 49z + 27}{z^3 + 5z^2 + 8z + 4}
\]

**Step 1.** The first step to write \( \frac{N(z)}{D(z)} \) in the form

\[
\frac{N(z)}{D(z)} = P(z) + \frac{R(z)}{D(z)}
\]

with \( P(z) \) being a polynomial and \( R(z) \) being a polynomial of degree strictly smaller than the degree of \( D(z) \). By long division

\[
z^3 + 5z^2 + 8z + 4 \left\lfloor \frac{z + 4}{z^4 + 9z^3 + 31z^2 + 49z + 27} \right. = z + 4 + \frac{3z^2 + 13z + 11}{z^3 + 5z^2 + 8z + 4}
\]

so

\[
\frac{z^4 + 9z^3 + 31z^2 + 49z + 27}{z^3 + 5z^2 + 8z + 4} = z + 4 + \frac{3z^2 + 13z + 11}{z^3 + 5z^2 + 8z + 4}
\]

**Step 2.** The second step is to factorize \( D(z) = z^3 + 5z^2 + 8z + 4 \). Any integer root of \( D(z) \) must divide the constant term, 4, exactly. Only \( \pm1, \pm2, \pm4 \) can be integer roots of \( z^3 + 5z^2 + 8z + 4 \). We test to see if \( \pm1 \) are roots.

\[
D(1) = (1)^3 + 5(1)^2 + 8(1) + 4 \neq 0 \quad \Rightarrow \quad z = 1 \text{ is not a root}
\]

\[
D(-1) = (-1)^3 + 5(-1)^2 + 8(-1) + 4 = 0 \quad \Rightarrow \quad z = -1 \text{ is a root}
\]

So \((z + 1)\) must divide \( z^3 + 5z^2 + 8z + 4 \). By long division
\[
\frac{z^2 + 4z + 4}{z + 1} = \frac{z^3 + 5z^2 + 8z + 4}{z^3 + z^2} = \frac{4z^2 + 8z + 4}{4z^2 + 4z} = \frac{4z + 4}{4z + 4} = 1
\]

so
\[
z^3 + 5z^2 + 8z + 4 = (z + 1)(z^2 + 4z + 4) = (z + 1)(z + 2)(z + 2)
\]

This is the end of step 2. We now know
\[
\frac{z^4 + 9z^3 + 31z^2 + 49z + 27}{z^3 + 5z^2 + 8z + 4} = z + 4 + \frac{3z^2 + 13z + 11}{(z + 1)(z + 2)^2}
\]

**Step 3.** The third step is to write \(\frac{3z^2 + 13z + 11}{(z + 1)(z + 2)^2}\) in the form
\[
\frac{3z^2 + 13z + 11}{(z + 1)(z + 2)^2} = \frac{A}{z + 1} + \frac{B}{z + 2} + \frac{C}{(z + 2)^2}
\]

for some constants \(A, B\) and \(C\). To determine the values of the constants \(A, B, C\), we put the right hand side back over the common denominator \((z + 1)(z + 2)^2\).
\[
\frac{3z^2 + 13z + 11}{(z + 1)(z + 2)^2} = \frac{A}{z + 1} + \frac{B}{z + 2} + \frac{C}{(z + 2)^2} = \frac{A(z + 2)^2 + B(z + 1)(z + 2) + C(z + 1)}{(z + 1)(z + 2)^2}
\]

The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.
\[
3z^2 + 13z + 11 = A(z + 2)^2 + B(z + 1)(z + 2) + C(z + 1)
\]

This must be true for all values of \(z\). In particular, it must be true for \(z = -1\). When \(z = -1\), the factor \((z + 1)\) multiplying \(B\) and \(C\) is exactly zero. So \(B\) and \(C\) disappear from the equation, leaving us with an easy equation to solve for \(A\):
\[
3z^2 + 13z + 11 \bigg|_{z=-1} = A(z + 2)^2 \bigg|_{z=-1} + B(z + 1)(z + 2) \bigg|_{z=-1} + C(z + 1) \bigg|_{z=-1} \implies 1 = A
\]

Sub this value of \(A\) back in and simplify.
\[
3z^2 + 13z + 11 = (1)(z + 2)^2 + B(z + 1)(z + 2) + C(z + 1)
\]
\[
2z^2 + 9z + 7 = B(z + 1)(z + 2) + C(z + 1) = (zB + 2B + C)(z + 1)
\]
Since \((z + 1)\) is a factor on the right hand side, it must also be a factor on the left hand side.

\[(2z + 7)(z + 1) = (zB + 2B + C)(z + 1) \implies (2z + 7) = (zB + 2B + C)\]

For the coefficients of \(z\) to match, \(B\) must be 2. For the constant terms to match, \(2B + C\) must be 7, so \(C\) must be 3. Subbing into \(\frac{3z^2+13z+11}{(z+1)(z+2)^2} = \frac{A}{z+1} + \frac{B}{z+2} + \frac{C}{(z+2)^2}\), we now have

\[
\frac{z^4 + 9z^3 + 31z^2 + 49z + 27}{z^3 + 5z^2 + 8z + 4} = z + 4 + \frac{1}{z+1} + \frac{2}{z+2} + \frac{3}{(z+2)^2}
\]