# The Partial Fractions Decomposition

## The Simplest Case

In the most common partial fraction decomposition, we split up

$$\frac{N(z)}{(z-a_1)\times\cdots\times(z-a_d)}$$

into a sum of the form

$$\frac{A_1}{z-a_1}+\cdots+\frac{A_d}{z-a_d}$$

We now show that this decomposition can always be achieved, *under the assumptions* that the $a_i$'s are all different and $N(z)$ is a polynomial of degree at most $d-1$. To do so, we shall repeatedly apply the following Lemma. (The word Lemma just signifies that the result is not that important – it is only used as a tool to prove a more important result.)

**Lemma 1** *Let $N(z)$ and $D(z)$ be polynomials of degree $n$ and $d$ respectively, with $n \le d$. Suppose that $a$ is NOT a zero of $D(z)$. Then there is a polynomial $P(z)$ of degree $p < d$ and a number $A$ such that*

$$\frac{N(z)}{D(z)\,(z-a)} = \frac{P(z)}{D(z)} + \frac{A}{z-a}$$

**Proof:** To save writing, let $\tilde z = z-a$. Then $\tilde N(\tilde z) = N(\tilde z + a)$ and $\tilde D(\tilde z) = D(\tilde z + a)$ are again polynomials of degree $n$ and $d$ respectively, $\tilde D(0) = D(a) \ne 0$ and we have to find a polynomial $\tilde P(\tilde z)$ of degree $p < d$ and a number $A$ such that

$$\frac{\tilde N(\tilde z)}{\tilde D(\tilde z)\,\tilde z} = \frac{\tilde P(\tilde z)}{\tilde D(\tilde z)} + \frac{A}{\tilde z} = \frac{\tilde P(\tilde z)\tilde z + A\tilde D(\tilde z)}{\tilde D(\tilde z)\,\tilde z}$$

or equivalently, such that

$$\tilde P(\tilde z)\tilde z + A\tilde D(\tilde z) = \tilde N(\tilde z)$$

Now look at the polynomial on the left hand side. Every term in $\tilde P(\tilde z)\tilde z$, has at least one power of $\tilde z$. So the constant term on the left hand side is exactly the constant term in $A\tilde D(\tilde z)$, which is $A\tilde D(0)$. The constant term on the right hand side is $\tilde N(0)$. So the constant terms on the left and right hand sides are the same if we choose $A = \frac{\tilde N(0)}{\tilde D(0)}$. Recall that $\tilde D(0)$ cannot be zero. Now move $A\tilde D(\tilde z)$ to the right hand side.

$$\tilde P(\tilde z)\tilde z = \tilde N(\tilde z) - A\tilde D(\tilde z)$$

The constant terms in $\tilde N(\tilde z)$ and $A\tilde D(\tilde z)$ are the same, so the right hand side contains no constant term and the right hand side is of the form $\tilde N_1(\tilde z)\tilde z$. Since $\tilde N(\tilde z)$ is of degree at most $d$ and $A\tilde D(\tilde z)$ is of degree exactly $d$, $\tilde N_1$ is a polynomial of at most degree $d-1$. It now suffices to choose $\tilde P(\tilde z) = \tilde N_1(\tilde z)$. ∎

Now back to

$$\frac{N(z)}{(z-a_1)\times\cdots\times(z-a_d)}$$

Apply Lemma 1, with $D(z) = (z - a_2) \times \cdots \times (z - a_d)$ and $a = a_1$. It says

$$\frac{N(z)}{(z-a_1)\times\cdots\times(z-a_d)} = \frac{A_1}{z-a_1} + \frac{P(z)}{(z-a_2)\times\cdots\times(z-a_d)}$$

for some polynomial $P$ of degree at most $d - 2$ and some number $A_1$. Apply Lemma 1 a second time, with $D(z) = (z - a_3) \times \cdots \times (z - a_d)$, $N(z) = P(z)$ and $a = a_2$. It says

$$\frac{P(z)}{(z-a_2)\times\cdots\times(z-a_d)} = \frac{A_2}{z-a_2} + \frac{Q(z)}{(z-a_3)\times\cdots\times(z-a_d)}$$

for some polynomial $Q$ of degree at most $d - 3$ and some number $A_2$. At this stage, we know that

$$\frac{N(z)}{(z-a_1)\times\cdots\times(z-a_d)} = \frac{A_1}{z-a_1} + \frac{A_2}{z-a_2} + \frac{Q(z)}{(z-a_3)\times\cdots\times(z-a_d)}$$

If we just keep going, repeatedly applying Lemma 1, we eventually end up with

$$\frac{N(z)}{(z-a_1)\times\cdots\times(z-a_d)} = \frac{A_1}{z-a_1} + \cdots + \frac{A_d}{z-a_d}$$

**The general case with linear factors**

Now consider splitting

$$\frac{N(x)}{(z-a_1)^{n_1}\times\cdots\times(z-a_d)^{n_d}}$$

into a sum of the form

$$\left[\frac{A_{1,1}}{z-a_1} + \cdots + \frac{A_{1,n_1}}{(z-a_1)^{n_1}}\right] + \cdots + \left[\frac{A_{d,1}}{z-a_d} + \cdots + \frac{A_{d,n_d}}{(z-a_d)^{n_d}}\right]$$

Note that, as we allow ourselves to use complex roots, this is the general case. We now show that this decomposition can always be achieved, under the assumptions that the $a_i$'s are all different and $N(x)$ is a polynomial of degree at most $n_1 + \cdots + n_d - 1$. (We can always ensure that the degree of the numerator is strictly smaller that the degree of the denominator by first pulling off a polynomial.) To do so, we shall repeatedly apply the following Lemma.

**Lemma 2** *Let $N(z)$ and $D(z)$ be polynomials of degree $n$ and $d$ respectively, with $n < d+m$. Suppose that $a$ is NOT a zero of $D(z)$. Then there is a polynomial $P(z)$ of degree $p < d$ and numbers $A_1, \cdots, A_m$ such that*

$$\frac{N(z)}{D(z)\,(z-a)^m} = \frac{P(z)}{D(z)} + \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \cdots + \frac{A_m}{(z-a)^m}$$

**Proof:** To save writing, let $\tilde z = z - a$. Then $\tilde N(\tilde z) = N(\tilde z + a)$ and $\tilde D(\tilde z) = D(\tilde z + a)$ are polynomials of degree $n$ and $d$ respectively, $\tilde D(0) = D(a) \neq 0$ and we have to find a polynomial $\tilde P(\tilde z)$ of degree $p < d$ and numbers $A_1, \cdots, A_m$ such that

$$\begin{aligned} \tfrac{\tilde N(\tilde z)}{\tilde D(\tilde z)\,\tilde z^m} &= \tfrac{\tilde P(\tilde z)}{\tilde D(\tilde z)} + \tfrac{A_1}{\tilde z} + \tfrac{A_2}{\tilde z^2} + \cdots + \tfrac{A_m}{\tilde z^m} \\ &= \tfrac{\tilde P(\tilde z)\tilde z^m + A_1\tilde z^{m-1}\tilde D(\tilde z) + A_2\tilde z^{m-2}\tilde D(\tilde z) + \cdots + A_m\tilde D(\tilde z)}{\tilde D(\tilde z)\,\tilde z^m} \end{aligned}$$

or equivalently, such that

$$\tilde P(\tilde z)\tilde z^m + A_1\tilde z^{m-1}\tilde D(\tilde z) + A_2\tilde z^{m-2}\tilde D(\tilde z) + \cdots + A_{m-1}\tilde z\tilde D(\tilde z) + A_m\tilde D(\tilde z) = \tilde N(\tilde z)$$

Now look at the polynomial on the left hand side. Every single term on the left hand side, except for the very last one, $A_m\tilde D(\tilde z)$, has at least one power of $\tilde z$. So the constant term on the left hand side is exactly the constant term in $A_m\tilde D(\tilde z)$, which is $A_m\tilde D(0)$. The constant term on the right hand side is $\tilde N(0)$. So the constant terms on the left and right hand sides are the same if we choose $A_m = \frac{\tilde N(0)}{\tilde D(0)}$. Recall that $\tilde D(0) \neq 0$. Now move $A_m\tilde D(\tilde z)$ to the right hand side.

$$\tilde P(\tilde z)\tilde z^m + A_1\tilde z^{m-1}\tilde D(\tilde z) + A_2\tilde z^{m-2}\tilde D(\tilde z) + \cdots + A_{m-1}\tilde z\tilde D(\tilde z) = \tilde N(\tilde z) - A_m\tilde D(\tilde z)$$

The constant terms in $\tilde N(\tilde z)$ and $A_m\tilde D(\tilde z)$ are the same, so the right hand side contains no constant term and is of the form $\tilde N_1(\tilde z)\tilde z$ with $\tilde N_1$ a polynomial of degree at most $d+m-2$. (Recall that $\tilde N$ is of degree at most $d+m-1$ and $\tilde D$ is of degree at most $d$.) Divide the whole equation by $\tilde z$.

$$\tilde P(\tilde z)\tilde z^{m-1} + A_1\tilde z^{m-2}\tilde D(\tilde z) + A_2\tilde z^{m-3}\tilde D(\tilde z) + \cdots + A_{m-1}\tilde D(\tilde z) = \tilde N_1(\tilde z)$$

Now, we can repeat the previous argument. The constant term on the left hand side, which is exactly $A_{m-1}\tilde D(0)$, matchs the constant term on the right hand side, which is $\tilde N_1(0)$, if we choose $A_{m-1} = \frac{\tilde N_1(0)}{\tilde D(0)}$. With this choice of $A_{m-1}$,

$$\tilde P(\tilde z)\tilde z^{m-1} + A_1\tilde z^{m-2}\tilde D(\tilde z) + A_2\tilde z^{m-3}\tilde D(\tilde z) + \cdots + A_{m-2}\tilde z\tilde D(\tilde z) = \tilde N_1(\tilde z) - A_{m-1}\tilde D(\tilde z) = \tilde N_2(\tilde z)\tilde z$$

with $\tilde N_2$ a polynomial of degree at most $d+m-3$. Divide by $\tilde z$ and continue. After $m$ steps like this, we end up with

$$\tilde P(\tilde z)\tilde z = \tilde N_{m-1}(\tilde z) - A_1\tilde D(\tilde z)$$

after having chosen $A_1 = \frac{\tilde N_{m-1}(0)}{\tilde D(0)}$. There is no constant term on the right side so that $\tilde N_{m-1}(\tilde z) - A_1\tilde D(\tilde z)$ is of the form $\tilde N_m(\tilde z)\tilde z$ with $\tilde N_m$ a polynomial of degree $d-1$. Choosing $\tilde P(\tilde z) = \tilde N_m(\tilde z)$ completes the proof. ■

 

Now back to

$$\frac{N(z)}{(z-a_1)^{n_1}\times\cdots\times(z-a_d)^{n_d}}$$

Apply Lemma 2, with $D(z) = (z-a_2)^{n_2} \times \cdots \times (z-a_d)^{n_d}$, $m = n_1$ and $a = a_1$. It says

$$\frac{N(z)}{(z-a_1)^{n_1}\times\cdots\times(z-a_d)^{n_d}} = \frac{A_{1,1}}{z-a_1} + \frac{A_{1,2}}{(z-a_1)^2} + \cdots + \frac{A_{1,n_1}}{(z-a)^{n_1}} + \frac{P(z)}{(z-a_2)^{n_2}\times\cdots\times(z-a_d)^{n_d}}$$

Apply Lemma 2 a second time, with $D(z) = (z-a_3)^{n_3} \times \cdots \times (z-a_d)^{n_d}$, $N(z) = P(z)$, $m = n_2$ and $a = a_2$. And so on. Eventually, we end up with

$$\left[\frac{A_{1,1}}{z-a_1} + \cdots + \frac{A_{1,n_1}}{(z-a_1)^{n_1}}\right] + \cdots + \left[\frac{A_{d,1}}{z-a_d} + \cdots + \frac{A_{d,n_d}}{(z-a_d)^{n_d}}\right]$$