Complex Numbers and Exponentials

Definition and Basic Operations

A complex number is nothing more than a point in the xy–plane. The sum and product of two complex numbers (x_1, y_1) and (x_2, y_2) is defined by

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2)\\ (x_1, y_1)\,(x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\end{aligned}$$

respectively. It is conventional to use the notation $x+iy$ (or in electrical engineering country $x+jy$) to stand for the complex number (x, y). In other words, it is conventional to write x in place of $(x, 0)$ and i in place of $(0, 1)$. In this notation, the sum and product of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is given by

$$\begin{aligned}z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2)\\ z_1z_2 &= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)\end{aligned}$$

The complex number i has the special property

$$i^2 = (0 + 1i)(0 + 1i) = (0 \times 0 - 1 \times 1) + i(0 \times 1 + 1 \times 0) = -1$$

For example, if $z = 1 + 2i$ and $w = 3 + 4i$, then

$$\begin{aligned}z + w &= (1 + 2i) + (3 + 4i) = 4 + 6i\\ zw &= (1 + 2i)(3 + 4i) \quad = 3 + 4i + 6i + 8i^2 = 3 + 4i + 6i - 8 = -5 + 10i\end{aligned}$$

Addition and multiplication of complex numbers obey the familiar algebraic rules

$$\begin{aligned}z_1 + z_2 &= z_2 + z_1 & z_1z_2 &= z_2z_1\\ z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 & z_1(z_2z_3) &= (z_1z_2)z_3\\ 0 + z_1 &= z_1 & 1z_1 &= z_1\\ z_1(z_2 + z_3) &= z_1z_2 + z_1z_3 & (z_1 + z_2)z_3 &= z_1z_3 + z_2z_3\end{aligned}$$

The negative of any complex number $z = x + iy$ is defined by $-z = -x + (-y)i$, and obeys $z + (-z) = 0$.

Other Operations

The complex conjugate of z is denoted $\bar{z}$ and is defined to be $\bar{z} = x - iy$. That is, to take the complex conjugate, one replaces every i by $-i$. Note that

$$z\bar{z} = (x + iy)(x - iy) = x^2 - ixy + ixy + y^2 = x^2 + y^2$$

is always a positive real number. In fact, it is the square of the distance from $x + iy$ (recall that this is the point (x, y) in the xy–plane) to 0 (which is the point $(0, 0)$). The distance from $z = x + iy$ to 0 is denoted $|z|$ and is called the absolute value, or modulus, of z. It is given by

$$|z| \;=\; \sqrt{x^2 + y^2} \;=\; \sqrt{z\bar{z}}$$

Since $z_1z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$,

$$\begin{aligned}|z_1z_2| &= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2}\\ &= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1y_2x_2y_1 + x_2^2y_1^2}\\ &= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}\\ &= |z_1||z_2|\end{aligned}$$

for all complex numbers z_1, z_2.

Since $|z|^2 = z\bar{z}$, we have $z\left(\frac{\bar{z}}{|z|^2}\right) = 1$ for all complex numbers $z \neq 0$. This says that the multiplicative inverse, denoted z^{-1} or $\frac{1}{z}$, of any nonzero complex number $z = x + iy$ is

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$$

It is easy to divide a complex number by a real number. For example

$$\frac{11+2i}{25} = \frac{11}{25} + \frac{2}{25}i$$

In general, there is a trick for rewriting any ratio of complex numbers as a ratio with a real denominator. For example, suppose that we want to find $\frac{1+2i}{3+4i}$. The trick is to multiply by $1 = \frac{3-4i}{3-4i}$. The number $3 - 4i$ is the complex conjugate of $3 + 4i$. Since $(3 + 4i)(3 - 4i) = 9 - 12i + 12i + 16 = 25$

$$\frac{1+2i}{3+4i} = \frac{1+2i}{3+4i}\frac{3-4i}{3-4i} = \frac{(1+2i)(3-4i)}{25} = \frac{11+2i}{25} = \frac{11}{25} + \frac{2}{25}i$$

The notations $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ stand for the real and imaginary parts of the complex number z, respectively. If $z = x + iy$ (with x and y real) they are defined by

$$\mathrm{Re}\, z = x \qquad \mathrm{Im}\, z = y$$

Note that both $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ are real numbers. Just subbing in $\bar{z} = x - iy$ gives

$$\mathrm{Re}\, z = \tfrac{1}{2}(z + \bar{z}) \qquad \mathrm{Im}\, z = \tfrac{1}{2i}(z - \bar{z})$$

The Complex Exponential

Definition and Basic Properties. For any complex number $z = x + iy$ the exponential e^z, is defined by

$$e^{x+iy} = e^x \cos y + ie^x \sin y$$

In particular, $e^{iy} = \cos y + i \sin y$. This definition is not as mysterious as it looks. We could also define e^{iy} by the subbing x by iy in the Taylor series expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \frac{(iy)^6}{6!} + \cdots$$

The even terms in this expansion are

$$1 + \frac{(iy)^2}{2!} + \frac{(iy)^4}{4!} + \frac{(iy)^6}{6!} + \cdots = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots = \cos y$$

and the odd terms in this expansion are

$$iy + \tfrac{(iy)^3}{3!} + \tfrac{(iy)^5}{5!} + \cdots = i\Big(y - \tfrac{y^3}{3!} + \tfrac{y^5}{5!} + \cdots\Big) = i\sin y$$

For any two complex numbers z_1 and z_2

$$\begin{aligned}
e^{z_1}e^{z_2} &= e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2)\\
&= e^{x_1+x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2)\\
&= e^{x_1+x_2}\left\{(\cos y_1\cos y_2 - \sin y_1\sin y_2) + i(\cos y_1\sin y_2 + \cos y_2\sin y_1)\right\}\\
&= e^{x_1+x_2}\left\{\cos(y_1+y_2) + i\sin(y_1+y_2)\right\}\\
&= e^{(x_1+x_2)+i(y_1+y_2)}\\
&= e^{z_1+z_2}
\end{aligned}$$

so that the familiar multiplication formula also applies to complex exponentials. For any complex number $c = \alpha + i\beta$ and real number t

$$e^{ct} = e^{\alpha t + i\beta t} = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)]$$

so that the derivative with respect to t

$$\begin{aligned}
\tfrac{d}{dt}e^{ct} &= \alpha e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] + e^{\alpha t}[-\beta\sin(\beta t) + i\beta\cos(\beta t)]\\
&= (\alpha + i\beta)e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)]\\
&= ce^{ct}
\end{aligned}$$

is also the familiar one.

Relationship with $\sin$ **and** $\cos$**.** When θ is a real number

$$\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta\\
e^{-i\theta} &= \cos\theta - i\sin\theta = \overline{e^{i\theta}}
\end{aligned}$$

are complex numbers of modulus one. Solving for $\cos\theta$ and $\sin\theta$ (by adding and subtracting the two equations)

$$\begin{aligned}
\cos\theta &= \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \ = \operatorname{Re} e^{i\theta}\\
\sin\theta &= \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \operatorname{Im} e^{i\theta}
\end{aligned}$$

These formulae make it easy derive trig identities. For example

$$\begin{aligned}
\cos\theta\cos\phi &= \tfrac{1}{4}(e^{i\theta} + e^{-i\theta})(e^{i\phi} + e^{-i\phi})\\
&= \tfrac{1}{4}(e^{i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)} + e^{-i(\theta+\phi)})\\
&= \tfrac{1}{4}(e^{i(\theta+\phi)} + e^{-i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)})\\
&= \tfrac{1}{2}\big(\cos(\theta+\phi) + \cos(\theta-\phi)\big)
\end{aligned}$$

and, using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$,

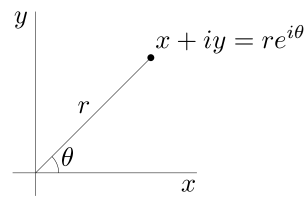
$$\begin{aligned}
\sin^3\theta &= -\tfrac{1}{8i}\big(e^{i\theta} - e^{-i\theta}\big)^3\\
&= -\tfrac{1}{8i}\big(e^{i3\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-i3\theta}\big)\\
&= \tfrac{3}{4}\tfrac{1}{2i}\big(e^{i\theta} - e^{-i\theta}\big) - \tfrac{1}{4}\tfrac{1}{2i}\big(e^{i3\theta} - e^{-i3\theta}\big)\\
&= \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin(3\theta)
\end{aligned}$$

and

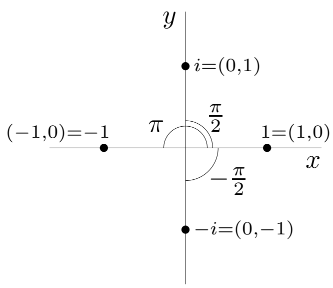
$$\begin{aligned}\cos(2\theta) &= \operatorname{Re} e^{i2\theta} = \operatorname{Re}\left(e^{i\theta}\right)^2 \\ &= \operatorname{Re}\left(\cos\theta + i\sin\theta\right)^2 \\ &= \operatorname{Re}\left(\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta\right) \\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Polar Coordinates. Let $z = x + iy$ be any complex number. Writing (x, y) in polar coordinates in the usual way gives $x = r\cos\theta$, $y = r\sin\theta$ and

$$x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$$



In particular



$$\begin{array}{rcllll} 1 &=& e^{i0} &= e^{2\pi i} &= e^{2k\pi i} & \text{for } k = 0, \pm1, \pm2, \cdots \\ -1 &=& e^{i\pi} &= e^{3\pi i} &= e^{(1+2k)\pi i} & \text{for } k = 0, \pm1, \pm2, \cdots \\ i &=& e^{i\pi/2} &= e^{\frac{5}{2}\pi i} &= e^{(\frac{1}{2}+2k)\pi i} & \text{for } k = 0, \pm1, \pm2, \cdots \\ -i &=& e^{-i\pi/2} &= e^{\frac{3}{2}\pi i} &= e^{(-\frac{1}{2}+2k)\pi i} & \text{for } k = 0, \pm1, \pm2, \cdots \end{array}$$

The polar coordinate $\theta = \tan^{-1}\frac{y}{x}$ associated with the complex number $z = x + iy$ is also called the argument of z.

The polar coordinate representation makes it easy to find square roots, third roots and so on. Fix any positive integer n. The n^{th} roots of unity are, by definition, all solutions z of

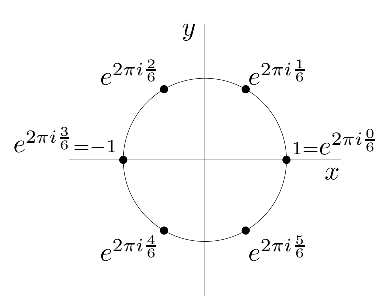
$$z^n = 1$$

Writing $z = re^{i\theta}$

$$r^n e^{n\theta i} = 1e^{0i}$$

The polar coordinates (r, θ) and (r', θ') represent the same point in the xy–plane if and only if $r = r'$ and $\theta = \theta' + 2k\pi$ for some integer k. So $z^n = 1$ if and only if $r^n = 1$, i.e. $r = 1$, and $n\theta = 2k\pi$ for some integer k. The n^{th} roots of unity are all complex numbers $e^{2\pi i\frac{k}{n}}$ with k integer. There are precisely n distinct n^{th} roots of unity because $e^{2\pi i\frac{k}{n}} = e^{2\pi i\frac{k'}{n}}$ if and only if $2\pi\frac{k}{n} - 2\pi i\frac{k'}{n} = 2\pi\frac{k-k'}{n}$ is an integer multiple of 2π. That is, if and only if $k - k'$ is an integer multiple of n. The n distinct nth roots of unity are

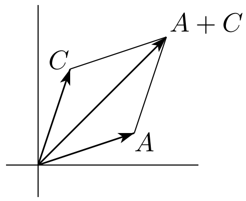
$$1\ ,\ e^{2\pi i\frac{1}{n}}\ ,\ e^{2\pi i\frac{2}{n}}\ ,\ e^{2\pi i\frac{3}{n}}\ ,\ \cdots\ ,\ e^{2\pi i\frac{n-1}{n}}$$

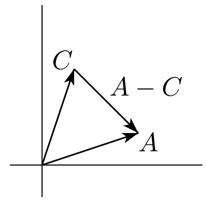


Sketching Complex Numbers as Vectors

Algebraic expressions involving complex numbers may be evaluated geometrically by exploiting the following two observations.

- (Addition and subtraction) A complex number is nothing more than a point in the xy–plane. So we may identify the complex number $A = a + ib$ with the vector whose tail is at the origin and whose head is at the point (a, b). Similarly, we may identify the complex number $C = c + id$ with the vector whose tail is at the origin and whose head is at the point (c, d). Those two vectors form two sides of a parallelogram. The vector for the sum $A+C = (a+c)+i(b+d)$ is that from the origin to the diagonally opposite corner of the parallelogram. The vector for the difference $A - C = (a - c) + i(b - d)$ has its tail at C and its head at A.





- (Multiplication and Division) To multiply or divide two complex numbers, write them in their polar coordinate forms $A = re^{i\theta}$, $C = \rho e^{i\varphi}$. So r and ρ are the lengths of A and C, respectively, and θ and φ are the angles from the positive x–axis to A and C, respectively. Then $AC = r\rho e^{i(\theta+\varphi)}$. This vector has length equal to the product of the lengths of A and C. The angle from the positive x–axis to AC is the sum of the angles θ and φ. And $\frac{A}{C} = \frac{r}{\rho} e^{i(\theta-\varphi)}$. This vector has length equal to the ratio of the lengths of A and C. The angle from the positive x–axis to AC is the difference of the angles θ and φ.