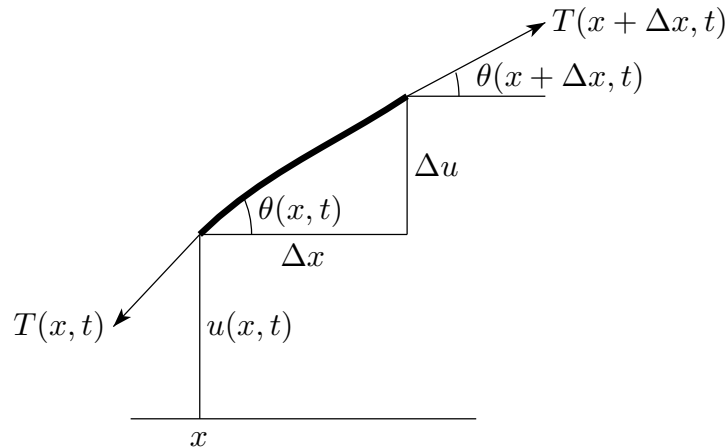


# Derivation of the Wave Equation

In these notes we apply Newton's law to an elastic string, concluding that small amplitude transverse vibrations of the string obey the wave equation. Consider a tiny element of the string.



The basic notation is

$u(x, t)$  = vertical displacement of the string from the  $x$  axis at position  $x$  and time  $t$

$\theta(x, t)$  = angle between the string and a horizontal line at position  $x$  and time  $t$

$T(x, t)$  = tension in the string at position  $x$  and time  $t$

$\rho(x)$  = mass density of the string at position  $x$

The forces acting on the tiny element of string at time  $t$  are

- tension pulling to the right, which has magnitude  $T(x + \Delta x, t)$  and acts at an angle  $\theta(x + \Delta x, t)$  above horizontal
- tension pulling to the left, which has magnitude  $T(x, t)$  and acts at an angle  $\theta(x, t)$  below horizontal and, possibly,
- various external forces, like gravity. We shall assume that all of the external forces act vertically and we shall denote by  $F(x, t)\Delta x$  the net magnitude of the external force acting on the element of string.

The mass of the element of string is essentially  $\rho(x)\sqrt{\Delta x^2 + \Delta u^2}$  so the vertical component of Newton's law says that

$$\rho(x)\sqrt{\Delta x^2 + \Delta u^2} \frac{\partial^2 u}{\partial t^2}(x, t) = T(x + \Delta x, t) \sin \theta(x + \Delta x, t) - T(x, t) \sin \theta(x, t) + F(x, t)\Delta x$$

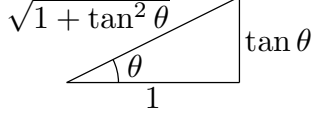
Dividing by  $\Delta x$  and taking the limit as  $\Delta x \rightarrow 0$  gives

$$\begin{aligned} \rho(x)\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} \frac{\partial^2 u}{\partial t^2}(x, t) &= \frac{\partial}{\partial x} [T(x, t) \sin \theta(x, t)] + F(x, t) \\ &= \frac{\partial T}{\partial x}(x, t) \sin \theta(x, t) + T(x, t) \cos \theta(x, t) \frac{\partial \theta}{\partial x}(x, t) + F(x, t) \end{aligned} \quad (1)$$

We can dispose of all the  $\theta$ 's by observing from the figure above that

$$\tan \theta(x, t) = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = \frac{\partial u}{\partial x}(x, t)$$

which implies, using the figure on the right below, that

$$\sin \theta(x, t) = \frac{\frac{\partial u}{\partial x}(x, t)}{\sqrt{1 + \left(\frac{\partial u}{\partial x}(x, t)\right)^2}} \quad \cos \theta(x, t) = \frac{1}{\sqrt{1 + \left(\frac{\partial u}{\partial x}(x, t)\right)^2}}$$


$$\theta(x, t) = \tan^{-1} \frac{\partial u}{\partial x}(x, t) \quad \frac{\partial \theta}{\partial x}(x, t) = \frac{\frac{\partial^2 u}{\partial x^2}(x, t)}{1 + \left(\frac{\partial u}{\partial x}(x, t)\right)^2}$$

Substituting these formulae into (1) give a horrendous mess. However, we can get considerable simplification by looking only at small vibrations. By a small vibration, we mean that  $|\theta(x, t)| \ll 1$  for all  $x$  and  $t$ . This implies that  $|\tan \theta(x, t)| \ll 1$ , hence that  $\left|\frac{\partial u}{\partial x}(x, t)\right| \ll 1$  and hence that

$$\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} \approx 1 \quad \sin \theta(x, t) \approx \frac{\partial u}{\partial x}(x, t) \quad \cos \theta(x, t) \approx 1 \quad \frac{\partial \theta}{\partial x}(x, t) \approx \frac{\partial^2 u}{\partial x^2}(x, t) \quad (2)$$

Substituting these into equation (1) give

$$\rho(x) \frac{\partial^2 u}{\partial t^2}(x, t) = \frac{\partial T}{\partial x}(x, t) \frac{\partial u}{\partial x}(x, t) + T(x, t) \frac{\partial^2 u}{\partial x^2}(x, t) + F(x, t) \quad (3)$$

which is indeed relatively simple, but still exhibits a problem. This is one equation in the two unknowns  $u$  and  $T$ .

Fortunately there is a second equation lurking in the background, that we haven't used yet. Namely, the horizontal component of Newton's law of motion. As a second simplification, we assume that there are only transverse vibrations. That is, our tiny string element moves only vertically. Then the net horizontal force on it must be zero. That is,

$$T(x + \Delta x, t) \cos \theta(x + \Delta x, t) - T(x, t) \cos \theta(x, t) = 0$$

Dividing by  $\Delta x$  and taking the limit as  $\Delta x$  tends to zero gives

$$\frac{\partial}{\partial x} [T(x, t) \cos \theta(x, t)] = 0$$

Thus  $T(x, t) \cos \theta(x, t)$  is independent of  $x$ . For small amplitude vibrations,  $\cos \theta$  is very close to one, for all  $x$ . So  $T$  is a function of  $t$  only, which is determined by how hard you are pulling on the ends of the string at time  $t$ . So for small, transverse vibrations, (3) simplifies further to

$$\rho(x) \frac{\partial^2 u}{\partial t^2}(x, t) = T(t) \frac{\partial^2 u}{\partial x^2}(x, t) + F(x, t) \quad (4)$$

In the event that the string density  $\rho$  is a constant, independent of  $x$ , the string tension  $T(t)$  is a constant independent of  $t$  (in other words you are not continually playing with the tuning pegs) and there are no external forces  $F$  we end up with the wave equation

$$\boxed{\frac{\partial^2 u}{\partial t^2}(x, t) = c^2 \frac{\partial^2 u}{\partial x^2}(x, t)} \quad \text{where} \quad c = \sqrt{\frac{T}{\rho}}$$