Using the Fourier Transform to Solve PDEs

In these notes we are going to solve the wave and telegraph equations on the full real line by Fourier transforming in the spatial variable. We start with

The Wave Equation

If $u(x,t)$ is the displacement from equilibrium of a string at position x and time t and if the string is undergoing small amplitude transverse vibrations, then we have seen that

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2\,\frac{\partial^2 u}{\partial x^2}(x,t) \tag{1}$$

for a constant c. We are now going to solve this equation by multiplying both sides by e^{-ikx} and integrating with respect to x. That is, we shall Fourier transform with respect to the spatial variable x. Denote the Fourier transform with respect to x, for each fixed t, of $u(x,t)$ by

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\ dx$$

We have already seen (in property (D) in the notes "Fourier Transforms") that the Fourier transform of the derivative $f'(x)$ is

$$\int_{-\infty}^{\infty} f'(x)e^{-ikx}\ dx = ik\int_{-\infty}^{\infty} f(x)e^{-ikx}\ dx = ik\hat{f}(k) \tag{2}$$

(by integration by parts with $u = e^{-ikx}$, $dv = f'(x)\,dx$, $du = -ike^{-ikx}\,dx$, $v(x) = f(x)$ and assuming that $f(x) \to 0$ as $x \to \pm\infty$). Applying this with $f(x) = \frac{\partial}{\partial x}u(x,t)$ and a second time with $f(x) = u(x,t)$, gives that the Fourier transform of $\frac{\partial^2 u}{\partial x^2}(x,t)$ is $-k^2\hat{u}(k,t)$. Computation of the Fourier transform of $\frac{\partial^2 u}{\partial t^2}(x,t)$ is even easier. For the first t–derivative,

$$\begin{aligned}
\int_{-\infty}^{\infty} u_t(x,t)e^{-ikx}\ dx &= \int_{-\infty}^{\infty} \lim_{h\to 0} \tfrac{u(x,t+h)-u(x,t)}{h} e^{-ikx}\ dx \\
&= \lim_{h\to 0} \tfrac{1}{h}\Big[\int_{-\infty}^{\infty} u(x,t+h)e^{-ikx}\ dx - \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\ dx\Big] \\
&= \lim_{h\to 0} \tfrac{1}{h}\big[\hat{u}(k,t+h) - \hat{u}(k,t)\big] \\
&= \tfrac{\partial}{\partial t}\hat{u}(k,t)
\end{aligned} \tag{3}$$

To get two t-derivatives, we just apply this twice (with u replaced by u_t the first time)

$$\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2}(x,t)\,e^{-ikx}\ dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}u(x,t)\,e^{-ikx}\ dx = \frac{\partial^2}{\partial t^2}\hat{u}(k,t)$$

So applying the Fourier transform to both sides of (1) gives

$$\frac{\partial^2}{\partial t^2}\hat{u}(k,t) = -c^2k^2\hat{u}(k,t) \tag{4}$$

This has not yet led to the solution for $u(x,t)$ or $\hat{u}(k,t)$, but it has led to a considerable simplification. We now have, for each fixed k, a constant coefficient, homogeneous, second order ordinary differential equation for $\hat{u}(k,t)$.

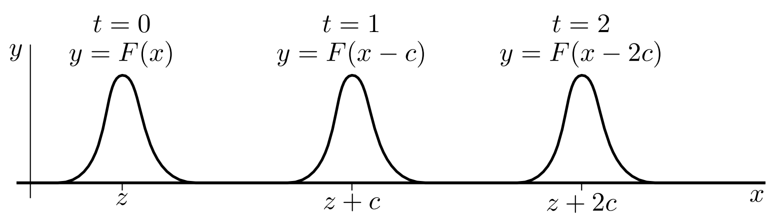
To emphasise that each k may now be treated independently, fix any k and write $\hat{u}(k,t) = U(t)$. The differential equation (4) now is $U''(t) + c^2k^2U(t) = 0$. From earlier courses, we know that this equation can be solved easily by trying $U(t) = e^{rt}$. Since $U''(t) + c^2k^2U(t) = (r^2 + c^2k^2)e^{rt} = 0$ if and only if $r = \pm ick$, the general solution to $U''(t)+c^2k^2U(t) = 0$, for any $k \neq 0$, is $U(t) = d_1e^{-ickt}+d_2e^{ickt}$. For $k = 0$, when the two values of $r = \pm ick$ are the same, the differential equation reduces to $U'' = 0$ and has general solution $U(t) = d_1 + d_2t$. We have to reject the d_2t solution (i.e. we have to require that $d_2 = 0$) on physical grounds – small transverse oscillations certainly do not include amplitudes that grow to infinity at t goes to infinity. Recalling that $U(t) = \hat{u}(k,t)$ we conclude that the general solution to (4) is

$$\hat{u}(k,t) = \hat{F}(k)e^{-ikct} + \hat{G}(k)e^{ikct}$$

We have renamed the arbitrary constants d_1 and d_2 to $\hat{F}(k)$ and $\hat{G}(k)$ respectively. The reason for these funny names will be made clear very soon. In any event, the arbitrary constants are certainly allowed to depend on k – viewed as an equation for an unknown function of t, (4) is a different equation for every different value of k. To recover $u(x,t)$ we just need to take the inverse Fourier transform

$$\begin{aligned}
u(x,t) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,t)e^{ikx}\ dk \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \big[\hat{F}(k)e^{-ikct} + \hat{G}(k)e^{ikct}\big]e^{ikx}\ dk \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k)e^{ik(x-ct)}\ dk + \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}(k)e^{ik(x+ct)}\ dk \\
&= F(x-ct) + G(x+ct)
\end{aligned}$$

This is called the D'Alembert form of the solution of the wave equation. The $F(x - ct)$ part of the solution represents a wave packet moving to the right with speed c. You can see this by observing that all points (x,t) in space time for which $x - ct$ takes the same fixed value, z, have the same value of $F(x - ct)$, namely $F(z)$. So if you move so that your position at time t is $x = z + ct$ (i.e. move the right with speed c) you always see the same string height. The figure below illustrates this. It contains the graphs of $F(x)$, $F(x-c) = F(x-ct)\big|_{t=1}$ and $F(x-2c) = F(x-ct)\big|_{t=2}$ for a bump shaped $F(x)$. In the figure I have chosen the location of the tick z on the x–axis so that $F(z) = \max_x F(x)$.



Similarly, $G(x + ct)$ represents a wave packet moving to the left with speed c.

Suppose, for example, the string starts at rest with the initial bump $u(x,0) = p(x)$. To satisfy these initial conditions, F and G must obey

$$p(x) = u(x,0) = F(x) + G(x)$$

and

$$0 = u_t(x,0) = -cF'(x) + cG'(x) \iff F'(x) = G'(x) \iff F(x) = G(x) + C$$

These equations only determine F and G up to an additive constant. This additive constant is irrelevant — adding any constant to F while subtracting the same constant from G does not change the value of

$F(x-ct)+G(x+ct)$ for any x or t. The functions $F(x)=G(x)=\frac{1}{2}p(x)$ do the job. So the bump resolves itself into two equal sized halves. One moves to the right with speed c and the other moves to the left with speed c. If the initial speed $u_t(x,0)=s(x)$ is not zero, the string behaves similarly, but the left and right moving parts need not have the same size and shape.

The Telegraph Equation

We may also use the same technique to solve the telegraph equation

$$u_{tt}+(\alpha+\beta)u_t+\alpha\beta u=c^2u_{xx} \tag{5}$$

though the details are somewhat messier. The case $\alpha=\beta=0$ is just the wave equation again. We now consider only the case that $\alpha,\beta>0$, i.e. that there is nonzero resistance in the wire and nonzero conductance to ground.

Again multiply both sides by e^{-ikx}, integrate with respect to x and denote the Fourier transform with respect to x, for each fixed t, of $u(x,t)$ by

$$\hat{u}(k,t)=\int_{-\infty}^{\infty}u(x,t)e^{-ikx}\,dx$$

Using (2) and (3) gives

$$\hat{u}_{tt}(k,t)+(\alpha+\beta)\hat{u}_t(k,t)+\alpha\beta\hat{u}(k,t)=-k^2c^2\hat{u}(k,t) \tag{6}$$

Once again, we have, for each fixed k, a constant coefficient, homogeneous, second order ordinary differential equation for $\hat{u}(k,t)$. To emphasise, again, that each k may now be treated independently, fix any k and write $\hat{u}(k,t)=U(t)$. The differential equation (6) now is

$$U''(t)+(\alpha+\beta)U'(t)+(\alpha\beta+c^2k^2)U(t)=0 \tag{6}$$

The guess $U(t)=e^{rt}$ is a solution if and only if

$$0=U''(t)+(\alpha+\beta)U'(t)+(\alpha\beta+c^2k^2)U(t)=\big(r^2+(\alpha+\beta)r+(\alpha\beta+c^2k^2)\big)e^{rt}=0$$

This is the case if and only if

$$r=\tfrac{-(\alpha+\beta)\pm\sqrt{(\alpha+\beta)^2-4(\alpha\beta+c^2k^2)}}{2}=\tfrac{-(\alpha+\beta)\pm\sqrt{(\alpha-\beta)^2-4c^2k^2}}{2}$$

If $4c^2k^2\le(\alpha-\beta)^2$ then both values of r are real and negative. To see this, note that

$$\begin{aligned}
&0\le(\alpha-\beta)^2-4c^2k^2\le(\alpha-\beta)^2\\
\Rightarrow\quad &0\le\sqrt{(\alpha-\beta)^2-4c^2k^2}\le|\alpha-\beta|\\
\Rightarrow\quad &\tfrac{-(\alpha+\beta)-|\alpha-\beta|}{2}\le\tfrac{-(\alpha+\beta)\pm\sqrt{(\alpha-\beta)^2-4c^2k^2}}{2}\le\tfrac{-(\alpha+\beta)+|\alpha-\beta|}{2}
\end{aligned}$$

Because $|\alpha-\beta|$ is either $\alpha-\beta$ (if $\alpha\ge\beta$) or $\beta-\alpha$ (if $\alpha\le\beta$), $\tfrac{-(\alpha+\beta)\pm\sqrt{(\alpha-\beta)^2-4c^2k^2}}{2}$ is a real number between $-\alpha$ and $-\beta$ and both values of r are strictly negative. Then both solutions for $U(t)$ damp to zero as $t\to\infty$ and thus represent transients in the wire.

So we shall only consider $\hat{u}(k,t)$'s that are zero unless $4c^2k^2>(\alpha-\beta)^2$. Then

$$r=\tfrac{-(\alpha+\beta)\pm\sqrt{(\alpha-\beta)^2-4c^2k^2}}{2}=-\tfrac{\alpha+\beta}{2}\pm\tfrac{1}{2}i\sqrt{4c^2k^2-(\alpha-\beta)^2}=-\gamma\pm i\omega(k)$$

where

$$\gamma = \tfrac{\alpha+\beta}{2},\ \omega(k) = \sqrt{c^2k^2 - \left(\tfrac{\alpha-\beta}{2}\right)^2}$$

so that the general solution to (6) is

$$\begin{aligned}\hat{u}(k,t) = U(t) &= \hat{F}(k)e^{[-\gamma-i\omega(k)]t} + \hat{G}(k)e^{[-\gamma+i\omega(k)]t}\\ &= e^{-\gamma t}\big[\hat{F}(k)e^{-i\omega(k)t} + \hat{G}(k)e^{i\omega(k)t}\big]\end{aligned}$$

where $\hat{F}(k)$ and $\hat{G}(k)$ are arbitrary constants (recall that k is a parameter which is just a constant as far as the ODE (6) is concerned). So

$$u(x,t) = \tfrac{1}{2\pi}e^{-\gamma t}\int_{-\infty}^{\infty}\big[\hat{F}(k)e^{-i\omega(k)t} + \hat{G}(k)e^{i\omega(k)t}\big]e^{ikx}\ dk$$

If we carefully tune our telegraph wire so that $\alpha = \beta$, then $\omega(k) = ck$ and

$$\begin{aligned}u(x,t) &= \tfrac{1}{2\pi}e^{-\gamma t}\int_{-\infty}^{\infty}\big[\hat{F}(k)e^{-ikct} + \hat{G}(k)e^{ikct}\big]e^{ikx}\ dk\\ &= \tfrac{1}{2\pi}e^{-\gamma t}\int_{-\infty}^{\infty}\big[\hat{F}(k)e^{ik(x-ct)} + \hat{G}(k)e^{ik(x+ct)}\big]\ dk\\ &= e^{-\gamma t}\big[F(x-ct) + G(x+ct)\big]\end{aligned}$$

As in the case of the solution to the wave equation, we have a wave packet that is moving to the right with speed c and a wave packet that is moving to the left with speed c. The wave packets do not change shape as time progresses, but the factor of $e^{-\gamma t}$ causes the size of the packets to diminish. If we put in amplifiers periodically along the wire, we can use it to transmit signals without distortion.

If $\alpha \neq \beta$, then $\omega(k)$ is a more complicated function of k. If $\hat{F}(k)$ is nonzero only for k in a narrow interval around some fixed (spatial frequency) k_0, then in that interval $\omega(k)$ is essentially given by the beginning of its Taylor expansion about k_0, which is $\omega(k_0) + \frac{d\omega}{dk}(k_0)\,(k-k_0)$. Writing $v(k_0) = \frac{d\omega}{dk}(k_0)$,

$$\begin{aligned}\tfrac{1}{2\pi}e^{-\gamma t}\int_{-\infty}^{\infty}\hat{F}(k)e^{-i\omega(k)t}e^{ikx}\ dk &= \tfrac{1}{2\pi}e^{-\gamma t}\int_{-\infty}^{\infty}\hat{F}(k)e^{-i[\omega(k_0)+v(k_0)(k-k_0)]t}e^{ikx}\ dk\\ &= e^{-\gamma t}e^{-i[\omega(k_0)-v(k_0)k_0]t}\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}(k)e^{ik[x-v(k_0)t]}\ dk\\ &= e^{-\gamma t}e^{-i[\omega(k_0)-v(k_0)k_0]t}F\big(x-v(k_0)t\big)\end{aligned}$$

This tells us that the part of the wave packet with k very close to k_0 travels with speed $v(k_0)$, which is called the group velocity. Except when $\alpha = \beta$, $v(k_0)$ has a nontrivial dependence on k_0. So parts of the wave packet with different values of k travel with different speeds. This is called dispersion. It causes a distortion of the wave packet, which is illustrated in an applet on our course web page.