Review of Ordinary Differential Equations

Definition 1

(a) A **differential equation** is an equation for an unknown function that contains the derivatives of that unknown function. For example $y''(t) + y(t) = 0$ is a differential equation for the unknown function $y(t)$.

(b) A differential equation is called an **ordinary differential equation** (often shortened to "ODE") if only ordinary derivatives appear. That is, if the unknown function has only a single independent variable. A differential equation is called a **partial differential equation** (often shortened to "PDE") if partial derivatives appear. That is, if the unknown function has more than one independent variable. For example $y''(t) + y(t) = 0$ is an ODE while $\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t)$ is a PDE.

(c) The **order** of a differential equation is the order of the highest derivative that appears. For example $y''(t) + y(t) = 0$ is a second order ODE.

(d) An ordinary differential equation that is of the form

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_{n-1}(t)y'(t) + a_n(t)y(t) = F(t) \tag{1}$$

with given coefficient functions $a_0(t)$, $\cdots$, $a_n(t)$ and $F(t)$ is said to be **linear**. Otherwise, the ODE is said to be **nonlinear**. For example, $y'(t)^2 + y(t) = 0$, $y'(t)y''(t) + y(t) = 0$ and $y'(t) = e^{y(t)}$ are all nonlinear.

(e) The ODE (1) is said to have **constant coefficients** if the coefficients $a_0(t)$, $a_1(t)$, $\cdots$, $a_n(t)$ are all contants. Otherwise, it is said to have **variable coefficients**. For example, the ODE $y''(t) + 7y(t) = \sin t$ is constant coefficient, while $y''(t) + ty(t) = \sin t$ is variable coefficient.

(f) The ODE (1) is said to be **homogeneous** if $F(t)$ is identically zero. Otherwise, it is said to be **inhomogeneous** or **nonhomogeneous**. For example, the ODE $y''(t) + 7y(t) = 0$ is homogeneous, while $y''(t) + 7y(t) = \sin t$ is inhomogeneous. A homogeneous ODE always has the trivial solution $y(t) = 0$.

(g) An **initial value problem** is a problem in which one is to find an unknown function $y(t)$ that satisfies both a given ODE and given initial conditions, like $y(0) = 1$, $y'(0) = 0$.

(h) A **boundary value problem** is a problem in which one is to find an unknown function $y(t)$ that satisfies both a given ODE and given boundary conditions, like $y(0) = 0$, $y(1) = 0$.

Theorem 2 *Assume that the coefficients $a_0(t)$, $a_1(t)$, $\cdots$, $a_{n-1}(t)$, $a_n(t)$ and $F(t)$ are reasonably smooth, bounded functions and that $a_0(t)$ is not zero.*

(a) The general solution to the ODE (1) is of the form

$$y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t) + \cdots + C_n y_n(t)$$

where

- *n is the order of the ODE (1)*
- *the particular solution, $y_p(t)$, is any solution to (1)*
- *C_1, C_2, $\cdots$, C_n are arbitrary constants*
- *y_1, y_2, $\cdots$, y_n are n independent solutions to the homogenous equation*

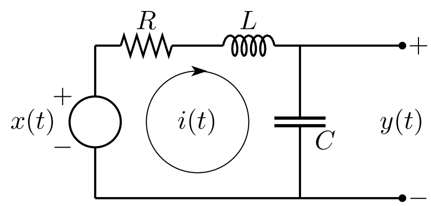
$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_{n-1}(t)y'(t) + a_n(t)y(t) = 0$$

"Independent" just means that no y_i can be written as a linear combination of the other y_j's. For example, $y_1(t)$ cannot be expressed in the form $d_2 y_2(t) + \cdots + d_n y_n(t)$.

(b) Given any constants $b_0, \cdots, b_{n-1}$ there is exactly one function $y(t)$ that obeys the ODE (1) and the initial conditions

$$y(0) = b_0 \qquad y'(0) = b_1 \qquad \cdots \qquad y^{(n-1)}(0) = b_{n-1}$$

Example 3 (The RLC circuit) As an example of the most commonly used techniques for solving ODE's, we consider the RLC circuit



We're going to think of the voltage $x(t)$ as a input signal and the voltage $y(t)$ as an output signal. The goal is to determine the output voltage for a given input voltage. In the notes "The RLC Circuit", we derived the ODE

$$LCy''(t) + RCy'(t) + y(t) = x(t) \tag{2}$$

As a concrete example, we'll take an ac voltage source and choose the origin of time so that $x(0) = 0$, $x(t) = E_0 \sin(\omega t)$. Then the differential equation becomes

$$LCy''(t) + RCy'(t) + y(t) = E_0 \sin(\omega t) \tag{3}$$

This is a second order, linear, constant coefficient ODE. So we know, from Theorem 2, that the general solution is of the form $y_p(t) + C_1 y_1(t) + C_2 y_2(t)$, where

- $y_p(t)$, the particular solution, is any one solution to (3),
- C_1, C_2 are arbitrary constants and
- $y_1(t)$, $y_2(t)$ are any two independent solutions of the corresponding homogeneous equation

$$LCy''(t) + RCy'(t) + y(t) = 0 \tag{$3_{\rm h}$}$$

So to find the general solution to (3), we need to find three functions: $y_1(t)$, $y_2(t)$ and $y_p(t)$.

Finding $y_1(t)$ and $y_2(t)$: The best way to find y_1 and y_2 is to guess them. Any solution, $y_h(t)$, of ($3_{\rm h}$) has to have the property that $y_h(t)$, $RCy_h'(t)$ and $LCy_h''(t)$ have to cancel each other out for all t. We choose our guess so that $y_h(t)$, $y_h'(t)$ and $y_h''(t)$ are all proportional to a single function of t. Then it will be easy to see if $y_h(t)$, $RCy_h'(t)$ and $LCy_h''(t)$ all cancel. Hence we try $y_h(t) = e^{rt}$, with the constant r to the determined. This guess is a solution of ($3_{\rm h}$) if and only if

$$LCr^2 e^{rt} + RCre^{rt} + e^{rt} = 0 \iff LCr^2 + RCr + 1 = 0 \iff r = \tfrac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} \equiv r_{1,2} \tag{4}$$

Finding $y_1(t)$ and $y_2(t)$, when $R^2C^2 - 4LC \ne 0$: In the event that $R^2C^2 - 4LC \ne 0$, that is $R \ne 2\sqrt{\frac{L}{C}}$, r_1 and r_2 are different and we may take $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.

Finding $y_1(t)$ and $y_2(t)$, when $R^2C^2 - 4LC = 0$: In the event that $R = 2\sqrt{\frac{L}{C}}$, $r_1 = r_2$. Then we may take $y_1 = e^{r_1 t}$, but $e^{r_2 t} = e^{r_1 t}$ is certainly not a second independent solution. So we still need to find y_2. Here is a trick (called reduction of order) for finding the other solutions: look for solutions of the form $v(t)e^{-r_1 t}$. Here $e^{-r_1 t}$ is the solution we have already found and $v(t)$ is to be determined. To save writing, set $\rho = \frac{R}{2L}$ so that $r_1 = r_2 = \rho$. To save writing also divide (3_h) by LC and substitute that $\frac{R}{L} = 2\rho$ and $\frac{1}{LC} = \frac{R^2}{4L^2} = \rho^2$ (recall that we are assuming that $R^2 = \frac{4L}{C}$). So (3_h) is equivalent to

$$y_h''(t) + 2\rho\, y_h'(t) + \rho^2\, y_h(t) = 0$$

Sub in

$$\begin{aligned} y_h(t) &= \qquad v(t)e^{-\rho t} \\ y_h'(t) &= -\rho\ v(t)e^{-\rho t} + \quad v'(t)e^{-\rho t} \\ y_h''(t) &= \quad \rho^2 v(t)e^{-\rho t} - 2\rho v'(t)e^{-\rho t} + v''(t)e^{-\rho t} \end{aligned}$$

Thus when $y_h(t) = v(t)e^{-\rho t}$,

$$\begin{aligned} y_h''(t) + 2\rho\, y_h'(t) + \rho^2\, y_h(t) &= \big[\rho^2 - 2\rho^2 + \rho^2\big] v(t)e^{-\rho t} + \big[-2\rho + 2\rho\big] v'(t)e^{-\rho t} + v''(t)e^{-\rho t} \\ &= v''(t)e^{-\rho t} \end{aligned}$$

Thus $v(t)e^{-\rho t}$ is a solution of (3_h) whenever the function $v''(t) = 0$ for all t. But, for any values of the constants C_1 and C_2, $v(t) = C_1 + C_2 t$ has vanishing second derivative so $\big(C_1 + C_2 t\big)e^{-\rho t} = \big(C_1 + C_2 t\big)e^{-r_1 t}$ solves (3_h). This is of the form $C_1 y_1(t) + C_2 y_2(t)$ with $y_1(t) = e^{-r_1 t}$, the solution that we found first, and $y_2(t) = te^{-r_1 t}$, a second independent solution. So we may take $y_2(t) = te^{r_1 t}$.

Finding $y_p(t)$: The best way to find y_p is to guess it. We guess that the circuit responds to an oscillating applied voltage with a current that oscillates at the same frequency. So we try $y_p(t) = A\sin(\omega t - \varphi)$ with the amplitude A and phase φ to be determined. For $y_p(t)$ to be a solution, we need

$$\begin{aligned} LCy_p''(t) + RCy_p'(t) + y_p(t) &= E_0 \sin(\omega t) \qquad (3_p) \\ -LC\omega^2 A\sin(\omega t - \varphi) + RC\omega A\cos(\omega t - \varphi) + A\sin(\omega t - \varphi) &= E_0 \sin(\omega t) \\ &= E_0 \sin(\omega t - \varphi + \varphi) \end{aligned}$$

and hence, applying $\sin(A + B) = \sin A\cos B + \cos A\sin B$ with $A = \omega t - \varphi$ and $B = \varphi$,

$$\big(1 - LC\omega^2\big)A\sin(\omega t - \varphi) + RC\omega A\cos(\omega t - \varphi) = E_0\cos(\varphi)\sin(\omega t - \varphi) + E_0\sin(\varphi)\cos(\omega t - \varphi)$$

Matching coefficients of $\sin(\omega t - \varphi)$ and $\cos(\omega t - \varphi)$ on the left and right hand sides gives

$$\big(1 - LC\omega^2\big)A = E_0\cos(\varphi) \qquad (5)$$

$$RC\omega A = E_0\sin(\varphi) \qquad (6)$$

It is now easy to solve for A and φ

$$\begin{aligned} \frac{(6)}{(5)} &\implies \tan(\varphi) = \frac{RC\omega}{1 - LC\omega^2} &&\implies \varphi = \tan^{-1}\frac{RC\omega}{1 - LC\omega^2} \\ \sqrt{(5)^2 + (6)^2} &\implies \sqrt{\big(1 - LC\omega^2\big)^2 + R^2C^2\omega^2}\ A = E_0 &&\implies A = \frac{E_0}{\sqrt{(1 - LC\omega^2)^2 + R^2C^2\omega^2}} \end{aligned} \qquad (7)$$

Example 4 (Boundary Value Problems) By part (b) of Theorem 2, an initial value problem consisting of an n^{th} order linear ODE with reasonable coefficients and n initial conditions always has exactly one solution. We shall now see that a boundary value problem may have no solutions at all. Or it may have exactly one solution. Or it may have infinitely many solutions. We shall start by finding all solutions to the ODE

$$y'' + y = 0 \tag{8}$$

We shall then impose boundary conditions.

The function $y(t) = e^{rt}$ is a solution to (8) if and only if

$$r^2 e^{rt} + e^{rt} = 0 \iff r^2 + 1 = 0 \iff r = \pm i$$

where i (which electrical engineers often denote j) is a square root of -1. Thus the general solution to the second order linear ODE (8) is $y(t) = C_1' e^{it} + C_2' e^{-it}$, with C_1' and C_2' arbitrary constants. We may rewrite this general solution in terms of $\sin t$ and $\cos t$ by substituting in

$$e^{it} = \cos t + i \sin t \qquad e^{-it} = \cos t - i \sin t$$

This gives

$$y(t) = C_1'\big(\cos t + i \sin t\big) + C_2'(\cos t - i \sin t) = C_1 \cos t + C_2 \sin t \text{ where } C_1 = C_1' + C_2',\ C_2 = i(C_1' - C_2')$$

Note that there is nothing stopping C_1' and C_2' from being complex numbers. So there is nothing stopping C_1 and C_2 being real numbers.

Example 4.a Now consider the boundary value problem

$$y'' + y = 0 \qquad y(0) = 0 \qquad y(2\pi) = 1 \tag{9}$$

The function $y(t)$ satisfies the ODE if and only if it is of the form $y(t) = C_1 \cos t + C_2 \sin t$ for some constants C_1 and C_2. A function of this form satisfies the boundary condition $y(0) = 0$ if and only if

$$0 = y(0) = C_1 \cos 0 + C_2 \sin 0 = C_1$$

A function of this form satisfies the boundary condition $y(2\pi) = 1$ if and only if

$$1 = y(2\pi) = C_1 \cos 2\pi + C_2 \sin 2\pi = C_1$$

The two requirements $C_1 = 0$ and $C_1 = 1$ are incompatible. So the boundary value problem (9) has no solution at all.

Example 4.b Next consider the boundary value problem

$$y'' + y = 0 \qquad y(0) = 0 \qquad y\big(\tfrac{\pi}{2}\big) = 0 \tag{10}$$

The function $y(t)$ satisfies the ODE if and only if it is of the form $y(t) = C_1 \cos t + C_2 \sin t$ for some constants C_1 and C_2. A function of this form satisfies the boundary condition $y(0) = 0$ if and only if

$$0 = y(0) = C_1 \cos 0 + C_2 \sin 0 = C_1$$

A function of this form satisfies the boundary condition $y\big(\frac{\pi}{2}\big) = 0$ if and only if

$$0 = y\big(\tfrac{\pi}{2}\big) = C_1 \cos\big(\tfrac{\pi}{2}\big) + C_2 \sin\big(\tfrac{\pi}{2}\big) = C_2$$

So we have a solution if and only if $C_1 = C_2 = 0$ and the boundary value problem (10) has exactly one solution, namely $y(t) = 0$.

Example 4.c Finally consider the boundary value problem

$$y'' + y = 0 \qquad y(0) = 0 \qquad y(2\pi) = 0 \tag{11}$$

The function $y(t)$ satisfies the ODE if and only if it is of the form $y(t) = C_1 \cos t + C_2 \sin t$ for some constants C_1 and C_2. A function of this form satisfies the boundary condition $y(0) = 0$ if and only if

$$0 = y(0) = C_1 \cos 0 + C_2 \sin 0 = C_1$$

A function of this form satisfies the boundary condition $y(2\pi) = 0$ if and only if

$$0 = y(2\pi) = C_1 \cos(2\pi) + C_2 \sin(2\pi) = C_1$$

So we have a solution if and only if $C_1 = 0$ and the boundary value problem (11) has infinitely many solutions, namely $y(t) = C_2 \sin t$ with C_2 being an arbitrary constant.