

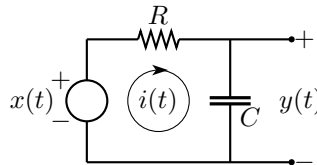
# Discrete-Time Linear, Time Invariant Systems and z-Transforms

## Linear, time invariant systems

“Continuous-time, linear, time invariant systems” refer to circuits or processors that take one input signal and produce one output signal with the following properties.

- Both the input and output are continuous-time signals.
- The system is linear. This means that if the input signals  $x_1(t)$  and  $x_2(t)$  generate the output signals  $y_1(t)$  and  $y_2(t)$ , respectively, and if  $a_1$  and  $a_2$  are constants, then the input signal  $a_1x_1(t) + a_2x_2(t)$  generates the output signal  $a_1y_1(t) + a_2y_2(t)$ .
- The system is time invariant. This means that if the input signal  $x(t)$  generates the output signal  $y(t)$ , then, for each real number  $s$ , the time shifted input signal  $\tilde{x}(t) = x(t - s)$  generates the time shifted output signal  $\tilde{y}(t) = y(t - s)$ .

**Example 1** A simple example of a continuous-time, linear, time invariant system is the RC lowpass filter that is used, for example in amplifiers, to suppress the high frequency parts of signals. This is the electrical circuit



with the voltage source  $x(t)$  viewed as the input signal and the voltage  $y(t)$  across the capacitor viewed as the output signal. If  $i(t)$  denotes the current in the circuit, the voltage across the resistor is  $Ri(t)$ . If  $q(t)$  denotes the charge on the capacitor, then the voltage across the capacitor is  $y(t) = \frac{q(t)}{C}$ . So by Kirchhoff's voltage law,

$$x(t) = Ri(t) + y(t)$$

Since  $i(t) = \frac{dq}{dt}(t) = C\frac{dy}{dt}(t)$ , we have that

$$RC\frac{dy}{dt}(t) + y(t) = x(t) \quad \Rightarrow \quad \frac{dy}{dt}(t) + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

This first order constant coefficient linear ODE is easily solved by multiplying it by the integrating factor<sup>(1)</sup>  $e^{t/RC}$ .

$$e^{t/RC}\frac{dy}{dt}(t) + \frac{1}{RC}e^{t/RC}y(t) = \frac{1}{RC}x(t)e^{t/RC} \quad \Rightarrow \quad \frac{d}{dt}(e^{t/RC}y(t)) = \frac{1}{RC}x(t)e^{t/RC}$$

Change  $t$  to  $\tau$  in this equation and integrate both sides from  $\tau = -\infty$  to  $\tau = t$ . Assuming that  $e^{t/RC}y(t)$  tends to zero as  $t \rightarrow -\infty$ ,

$$e^{t/RC}y(t) = \int_{-\infty}^t \frac{1}{RC}x(\tau)e^{\tau/RC}d\tau \quad \Rightarrow \quad y(t) = \int_{-\infty}^t \frac{1}{RC}x(\tau)e^{(\tau-t)/RC}d\tau$$

For each possible input signal  $x(t)$ , this determines the corresponding output signal  $y(t)$ . This rule is obviously linear. To see that it is also time invariant, observe that the output signal that corresponds to the

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<sup>(1)</sup> In general, multiplying the equation  $y'(t) + p(t)y(t) = g(t)$  by the integrating factor  $e^{\int p(t)dt}$  turns the left hand side into a perfect derivative.

input signal  $x(t - s)$  is

$$\int_{-\infty}^t \frac{1}{RC} x(\tau - s) e^{(\tau-t)/RC} d\tau \stackrel{\tau' = \tau - s}{=} \int_{-\infty}^{t-s} \frac{1}{RC} x(\tau') e^{(\tau'+s-t)/RC} d\tau' = y(t - s)$$

as desired.

“Discrete-time, linear, time invariant systems” refer to linear, time invariant circuits or processors that take one discrete-time input signal and produce one discrete-time output signal.

**Example 2** Let  $x[n]$  denote the net deposit (i.e. the sum of all deposits minus the sum of all withdrawals) to a bank account during month number  $n$  and let  $y[n]$  denote the balance in the account at the end of month number  $n$ . We want to think of  $x[n]$  as the known input to the system. We wish to determine the output  $y[n]$ . We start by setting up a equation that tells us about the time evolution of  $y[n]$ . At the end of month  $n - 1$ , the balance is  $y[n - 1]$ . During month  $n$  we add  $x[n]$  to the account. During this same month the bank adds some interest to the account. If the account pays  $r\%$  interest per month compounded monthly (say, for simplicity, on the balance at the end of the previous month), then the interest paid is  $\frac{r}{100}y[n - 1]$  so that the balance at the end of month number  $n$  is

$$y[n] = \left(1 + \frac{r}{100}\right)y[n - 1] + x[n] \quad (1)$$

If we started the account in month number  $N$ , we just make  $x[n] = y[n] = 0$  for all  $n < N$ . To find  $y[N]$ , we just need to set  $n = N$ :

$$y[N] = \left(1 + \frac{r}{100}\right)y[N - 1] + x[N] = x[N]$$

since  $y[N - 1] = 0$ . Then to find  $y[N + 1]$ , we just need to set  $n = N + 1$ :

$$y[N + 1] = \left(1 + \frac{r}{100}\right)y[N] + x[N + 1] = \left(1 + \frac{r}{100}\right)x[N] + x[N + 1]$$

Continuing in this way determines the output signal  $y[n]$  that corresponds to any given input signal  $x[n]$ . This is called iterating the equation (1).

**Example 3** A second way that discrete-time systems arise is through discrete-time approximations to continuous-time systems. Consider, for example, the  $RC$  circuit of Example 1. There, we saw that

$$\frac{dy}{dt}(t) + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

Of course this equation is so simple that we can easily solve it explicitly. But pretend that it is very complicated and we wish to use a computer to solve it. Computers can only store a finite number of pieces of data and can perform only a finite number of operations. So it is natural to approximate the continuous time equation that we are really interested in by an equation that only involves a discrete family of times, say separated by some fixed step size  $\Delta > 0$ . To do so we approximate

$$\frac{dy}{dt}(t) \approx \frac{y(t) - y(t - \Delta)}{\Delta}$$

and then the differential equation becomes

$$\frac{y(t) - y(t - \Delta)}{\Delta} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$



Our first deposit was in month number 0, so  $x[n] = h[n] = 0$  for all  $n < 0$ . Iterate (4), starting with  $n = 0$ .

$$\begin{aligned} n = 0 &\Rightarrow h[0] = ah[-1] + 1 = 1 \\ n = 1 &\Rightarrow h[1] = ah[0] = a \\ n = 2 &\Rightarrow h[2] = ah[1] = a^2 \\ n = 3 &\Rightarrow h[3] = ah[2] = a^3 \end{aligned}$$

and so on. The impulse response is

$$h[n] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ a^n & \text{if } n > 0 \end{cases} = a^n u[n] \quad \text{where } u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

For a general input  $x[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[k]a^{n-k}u[n-k]$$

Since  $u[n-k]$  takes the value 0 for  $n-k < 0$ , i.e.  $k > n$ , and the value 1 for  $n-k \geq 0$ , i.e.,  $k \leq n$ ,

$$y[n] = \sum_{k=-\infty}^n x[k]a^{n-k} = a^n \sum_{k=-\infty}^n a^{-k}x[k]$$

## Exponential signals and the $z$ -transform

The second important fact concerning the behaviour of discrete-time LTI systems is that all exponential signals are eigenfunctions for all LTI systems. Here is what that means. If  $z$  is any fixed complex number, then the signal  $x[n] = z^n$  is called an exponential signal. If we feed this exponential signal into a discrete-time LTI system with impulse response function  $h[n]$ , the output is

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = H(z)z^n$$

where

$$H(z) = \sum_{k=-\infty}^{\infty} z^{-k}h[k] \tag{5}$$

is called the  $z$ -transform of the impulse response  $h[n]$ . So when any exponential signal  $x[n] = z^n$  is fed into any LTI system, it is just multiplied by a constant (independent of time,  $n$ )  $H(z)$ . This multiplier,  $H(z)$  is called the eigenvalue of the eigenfunction  $x[n] = z^n$ .

The  $z$ -transform is in fact an extension of the discrete Fourier transform. If  $|z| = 1$ , then there is a real number  $\omega$  such that  $z = e^{i\omega}$  and

$$H(e^{i\omega}) = \sum_{k=-\infty}^{\infty} (e^{i\omega})^{-k}h[k] = \sum_{k=-\infty}^{\infty} e^{-i\omega k}h[k] = \hat{h}(\omega)$$

**Example 5** An amplifier is one of the simplest LTI systems. The output generated for the input  $x[n]$  is

$$y[n] = Kx[n]$$

where  $K$  is the amplification factor. In particular the impulse response function for this system is the output resulting from the input signal  $\delta[n]$ , which is

$$h[n] = K\delta[n] = \begin{cases} K & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

The  $z$ -transform of this impulse response function is

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k} = h[k]z^{-k}|_{k=0} = K$$

since  $h[k]$  takes a nonzero value only for  $k = 0$  and  $h[0] = K$ .

**Example 6** A second very simple LTI system is a delay line, which simply delays a signal by  $m$  units of time. The output generated for the input  $x[n]$  is

$$y[n] = x[n - m]$$

In particular, the impulse response function for this system is the output resulting from the input signal  $\delta[n]$ , which is

$$h[n] = \delta[n - m] = \delta_m[n] = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

The  $z$ -transform of this impulse response function is

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k} = h[k]z^{-k}|_{k=m} = z^{-m}$$

since  $h[k]$  takes a nonzero value only for  $k = m$  and  $h[m] = 1$ .

**Example 7** Consider once again the banking Examples 2 and 4. We saw that the impulse response is

$$h[n] = a^n u[n] = \begin{cases} 0 & \text{if } n < 0 \\ a^n & \text{if } n \geq 0 \end{cases}$$

It has  $z$ -transform

$$H(z) = \sum_{k=-\infty}^{\infty} z^{-k} h[k] = \sum_{k=0}^{\infty} z^{-k} a^k$$

Recalling that the geometric series  $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ , provided that  $|r| < 1$ ,

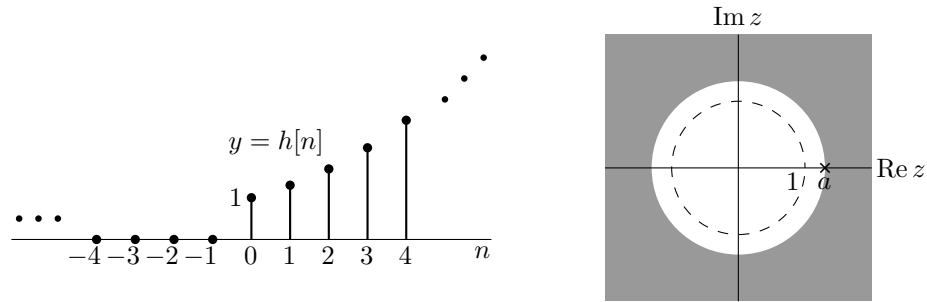
$$H(z) = \frac{1}{1-z^{-1}a} = \frac{z}{z-a}$$

provided that  $|z^{-1}a| < 1$ , or equivalently  $|z| > |a|$ . So, in this example, the sum defining the  $z$ -transform  $H(z)$  converges only for  $|z| > |a|$ . This region is called the *region of convergence*<sup>(2)</sup>, denoted ROC, of the  $z$ -transform. In this example  $H(z) = \frac{z}{z-a}$  has a pole (i.e. a singularity) at  $z = a$ . The figure on the left below is the graph of  $h[n]$ . The figure on the right shows the ROC and pole of  $H(z)$  in the case that  $a > 1$ .

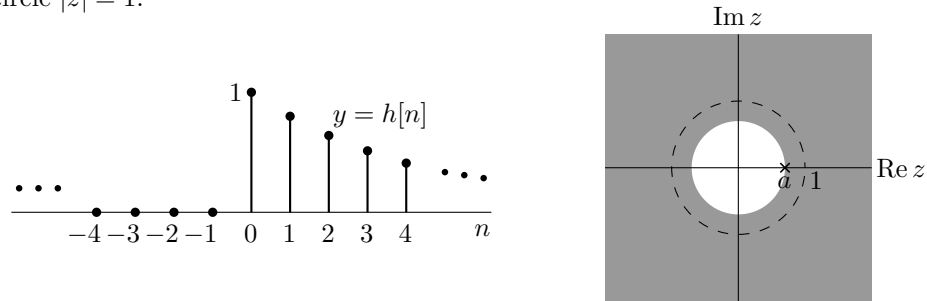
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<sup>(2)</sup> In general, the ROC for the  $z$ -transform of the impulse response function  $h[n]$  is the set of all complex numbers  $z$  such that  $\sum_{k=-\infty}^{\infty} |h[k]| |z|^{-k}$  is finite. Since this series depends only on  $|z|$ , the ROC is always a union of circles  $|z| = r$ .

The ROC is the shaded region. Note that, in this case,  $h[n]$  is growing exponentially as  $n \rightarrow \infty$  and the corresponding pole of  $H(z)$  is outside the unit circle  $|z| = 1$ .



In our banking example we had  $a > 1$ . In other applications it is perfectly possible to have  $a < 1$  or even complex  $a$ . The next pair of figures below show  $h[n]$  and the ROC and pole of  $H(z)$  in the case that  $0 < a < 1$ . In this case,  $h[n]$  is decaying exponentially as  $n \rightarrow \infty$  and the corresponding pole of  $H(z)$  is inside the unit circle  $|z| = 1$ .



**Example 8** Consider

$$h[n] = \begin{cases} 0 & \text{if } n \geq 0 \\ -b^n & \text{if } n < 0 \end{cases}$$

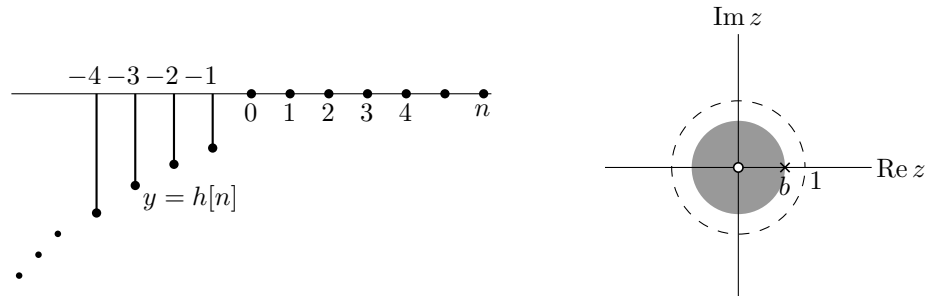
It has  $z$ -transform

$$H(z) = \sum_{k=-\infty}^{\infty} z^{-k} h[k] = - \sum_{k=-\infty}^{-1} z^{-k} b^k = - \sum_{m=1}^{\infty} z^m b^{-m} \text{ where } m = -k$$

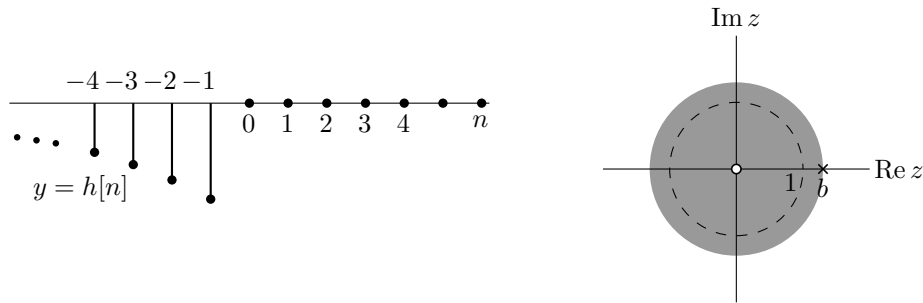
which is another geometric series with first term  $-\frac{z}{b}$  and ratio  $r = \frac{z}{b}$ . So it converges, if  $|\frac{z}{b}| < 1$ , or equivalently  $|z| < |b|$ , to

$$H(z) = -\frac{z/b}{1-z/b} = \frac{z}{z-b}$$

In this example we get  $H(z) = \frac{z}{z-b}$ , like the  $H(z)$  of Example 7, but with  $a$  replaced by  $b$ . It has a pole at  $z = b$  and a zero at  $z = 0$ . The locations of both zeros and poles of  $z$ -transforms are going to play big roles in determining the stability properties of LTI systems. More about this later. This time the ROC is inside, rather than outside, the circle containing the pole. Here are figures giving  $h[n]$  and the ROC, pole and zero of  $H(z)$  for  $b < 1$



and  $b > 1$



The ROC being inside the pole signals that  $h[n]$  is zero for all large positive  $n$ 's. Then, a pole outside of  $|z| = 1$  signals exponential decay while a pole inside of  $|z| = 1$  signals exponential growth, this time as  $n \rightarrow -\infty$ .

Note that if the  $a$  of Example 7 is equal to the  $b$  of Example 8, then the two examples give the identical algebraic formula for  $H(z)$ . The lesson we learn from this is that different  $h[n]$ 's can give the same algebraic formula for  $H(z)$ . The algebraic formula for  $H(z)$  does not, by itself, determine the impulse function  $h[n]$  uniquely. To recover  $h[n]$  we need some additional information, like the ROC. More about this later.

**Example 9** Now consider

$$h[n] = \begin{cases} a^n & \text{if } n \geq 0 \\ -b^n & \text{if } n < 0 \end{cases}$$

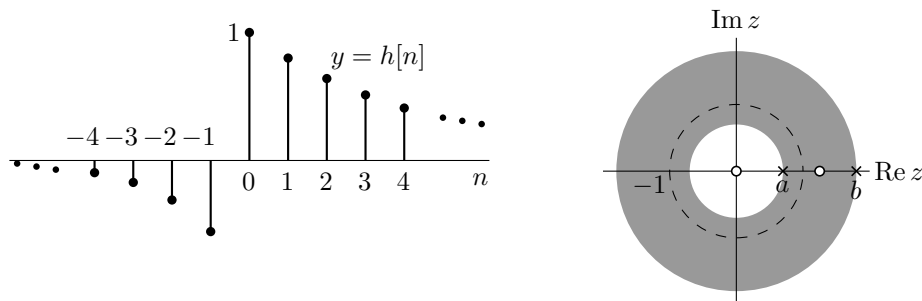
It has  $z$ -transform

$$H(z) = \sum_{k=-\infty}^{\infty} z^{-k} h[k] = - \sum_{k=-\infty}^{-1} z^{-k} b^k + \sum_{k=0}^{\infty} z^{-k} a^k$$

We have already ready found that the second sum converges to  $\frac{z}{z-a}$  for  $|z| > |a|$  and the first sum converges to  $\frac{z}{z-b}$  for  $|z| < |b|$ . So the two together converge to

$$H(z) = \frac{z}{z-a} + \frac{z}{z-b} = \frac{z(2z-a-b)}{(z-a)(z-b)}$$

for  $|a| < |z| < |b|$ . For this ROC to be nonempty we must have  $|a| < |b|$ . Here are figures giving  $h[n]$  and the ROC, poles and zeros of  $H(z)$  for  $0 < a < 1 < b$ .



## Causal systems

An additional common property of LTI systems is called causality. A causal system is one that does not respond until there is some input. In particular the response to a unit impulse may not start before  $n = 0$ . That is, an LTI system is **causal** if and only if its impulse response function obeys

$$h[n] = 0 \text{ for all } n < 0$$

For such a system, the output at time  $n$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^n x[k]h[n-k] \quad \text{since } h[n-k] = 0 \text{ for all } k > n$$

depends only on the input  $x[k]$  at times  $k \leq n$ . In particular, if the input  $x[n] = 0$  for all  $n < N$ , then the corresponding output automatically also vanishes for  $n < N$ . As we would expect, the banking system of Example 4 is causal.

For any causal LTI system

$$H(z) = \sum_{k=-\infty}^{\infty} z^{-k}h[k] = \sum_{k=0}^{\infty} z^{-k}h[k] \quad (6)$$

Then, as  $|z|$  increases, every single term in the sum gets smaller. Hence if the circle  $|z| = r_0$  is in the ROC, then the entire exterior  $|z| \geq r_0$  is automatically in the ROC as in Example 7. Furthermore, as  $|z| \rightarrow \infty$ , every term in (6) except for the  $k = 0$  term converges to zero so that

$$\lim_{|z| \rightarrow \infty} H(z) = h[0]$$

It is possible to prove that if the  $z$ -transform  $H(z)$  obeys

- if the circle  $|z| = r_0$  is in the ROC, then the entire exterior  $|z| \geq r_0$  is in the ROC and
- $\lim_{|z| \rightarrow \infty} H(z)$  exists

then there is a unique impulse response function  $h[n]$  whose  $z$ -transform is the given  $H(z)$  (with the same ROC) and furthermore  $h[n]$  is causal.

## Stable systems

By definition, an LTI system is **stable** if every bounded input necessarily generates a bounded output. In general, if a bounded input  $x[n]$ , obeying  $|x[n]| \leq B$ , is fed into an LTI with impulse response  $h[n]$ , the resulting output obeys

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| \leq \sum_{k=-\infty}^{\infty} |x[n-k]| |h[k]| \leq B \sum_{k=-\infty}^{\infty} |h[k]|$$

Hence any LTI system whose impulse response function is absolutely summable, i.e. whose impulse response function obeys  $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ , is stable. Conversely, if  $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$ , then the bounded input

$$x[n] = \begin{cases} \frac{\overline{h[-n]}}{|h[-n]|} & \text{if } h[-n] \neq 0 \\ 0 & \text{if } h[-n] = 0 \end{cases}$$

generates an output which at time 0 is

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k] = \sum_{\substack{k \text{ with} \\ h[-k] \neq 0}} \frac{\overline{h[-k]}}{|h[-k]|} h[-k] = \sum_{k=-\infty}^{\infty} |h[-k]| = \infty$$

So an LTI system is stable if and only if its impulse response function is absolutely summable. For  $|z| = 1$

$$\sum_{k=-\infty}^{\infty} |z^{-k}h[k]| = \sum_{k=-\infty}^{\infty} |h[k]|$$

so an LTI system is stable if and only if the circle  $|z| = 1$  is in the ROC for the  $z$ -transform of its impulse response function. A causal LTI system is stable if and only if its ROC includes all of  $|z| \geq 1$ . In particular a causal, stable LTI system has no poles in  $|z| \geq 1$ .

**Example 10** The ROC for the banking Example 4 is  $|z| > |a|$ . So this system is stable when  $|a| < 1$  and is unstable when  $|a| \geq 1$ . For a bank paying  $r\%$  interest per unit time, which is compounded at the end of each unit of time,  $a = 1 + \frac{r}{100}$ . If  $r > 0$  a single dollar deposited at time 0 generates an exponentially growing balance, resulting in instability. If  $r = 0$ , a deposit of one dollar per unit time (that's also a bounded input) generates a linearly growing balance, again resulting in instability. On the other hand if the bank charges you interest to store our money so that  $-100 < r < 0$ , then  $0 < a < 1$  and the system is stable.

### Recovering the impulse response function from its $z$ -transform

The  $z$ -transform is closely related to Fourier series and Fourier transforms. In fact, for  $z = e^{i\omega}$  (i.e. for  $z$  of modulus one)

$$H(e^{i\omega}) = \sum_{n=-\infty}^{\infty} e^{-i\omega n} h[n] = \hat{h}(\omega)$$

is exactly the Fourier transform  $\hat{h}(\omega)$  of the discrete-time signal  $h[n]$ , assuming that the ROC includes the circle  $|z| = 1$ . This tells us how to recover  $h[n]$  if we are told what  $H(z)$  is and if we are told that the ROC includes the circle  $|z| = 1$ :

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{h}(\omega) e^{i\omega n} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{i\omega}) e^{i\omega n} d\omega$$

In the, rather common, event that  $H(z)$  is a rational function, meaning that it is the ratio of two polynomials in  $z$ , there is a simpler technique that is based on the method of partial fractions from first year calculus. The idea is to write  $H(z)$  as a sum of simple terms that can be handled by table lookup. It is rare that one needs a larger table than

$z$ -Transform	Signal
$\frac{1}{z^m}$	$\delta_m[n] = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$
$\frac{1}{1-a/z}$	$e_a[n] = a^n u[n] = \begin{cases} a^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$

We saw these entries in Examples 6 and 7, respectively. Here we are assuming that the ROC is  $|z| > a$ , as is the case if the system is causal. If the ROC is  $|z| < a$ , then we use Example 8 instead. Because of the

prevalence of  $\frac{1}{z}$  in this table, it is convenient to make the change of variables  $\zeta = \frac{1}{z}$ . If  $H(z)$  is a rational function of  $z$ , then  $H(\frac{1}{\zeta})$  is a rational function of  $\zeta$ . In terms of  $\zeta$ , the lookup table is

$z$ -Transform	Signal
$\zeta^m$	$\delta_m[n] = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$
$\frac{1}{1-a\zeta}$	$e_a[n] = a^n u[n] \begin{cases} a^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$

Suppose that  $H(\frac{1}{\zeta}) = \frac{N(\zeta)}{D(\zeta)}$  with  $N$  and  $D$  polynomials. The first step in the method of partial fractions is to check that the degree of the numerator is strictly smaller than the degree of the denominator. If it isn't, use long division to rewrite

$$\frac{N(\zeta)}{D(\zeta)} = P(\zeta) + \frac{R(\zeta)}{D(\zeta)}$$

with  $P$  and  $R$  polynomials and with the degree of  $R$  strictly smaller than the degree of  $D$ . See the notes "Long Division" if you need a review.

The second step is to factor the denominator. Any polynomial,  $D(\zeta)$ , of degree  $p$  can be written as a product

$$D(\zeta) = a(\zeta - r_1) \cdots (\zeta - r_p)$$

of linear factors. The constant  $a$  is the coefficient of  $\zeta^p$  in  $D(\zeta)$  and the constants  $r_1, \dots, r_p$  are the roots of  $D(\zeta)$ . If the  $p$  roots are all different, which is the only case that we shall consider, then the rational function  $\frac{R(\zeta)}{D(\zeta)}$  can be written

$$\frac{R(\zeta)}{D(\zeta)} = \frac{c_1}{\zeta - r_1} + \frac{c_2}{\zeta - r_2} + \cdots + \frac{c_p}{\zeta - r_p} \quad (7)$$

A simple way to determine the coefficient, for example,  $c_2$  is to multiply both sides of (7) by  $\zeta - r_2$

$$(\zeta - r_2) \frac{R(\zeta)}{D(\zeta)} = c_1 \frac{\zeta - r_2}{\zeta - r_1} + c_2 + c_3 \frac{\zeta - r_2}{\zeta - r_3} + \cdots + c_p \frac{\zeta - r_2}{\zeta - r_p}$$

and then take the limit as  $\zeta \rightarrow r_2$ . All terms on the right hand side, except for  $c_2$  itself, then tend to zero, since we are assuming that  $r_j \neq r_2$  for all  $j \neq 2$ , so that

$$c_2 = \lim_{\zeta \rightarrow r_2} (\zeta - r_2) \frac{R(\zeta)}{D(\zeta)}$$

Both the numerator  $(\zeta - r_2)R(\zeta)$  and the denominator  $D(\zeta)$  tend to zero as  $\zeta \rightarrow r_2$ , so we have an indeterminate form. But the limit can be evaluated using l'Hôpital's rule.

$$c_2 = \lim_{\zeta \rightarrow r_2} \frac{R(\zeta) + (\zeta - r_2)R'(\zeta)}{D'(\zeta)} = \frac{R(r_2)}{D'(r_2)}$$

Equation (7) reduces the problem of finding the impulse response corresponding to  $\frac{R(\zeta)}{D(\zeta)}$  to the problem of finding the impulse response corresponding to

$$\frac{c}{\zeta - r} = -\frac{c}{r} \frac{1}{1 - \zeta/r}$$

which, by our second table entry, is  $-\frac{c}{r} e_{1/r}[n]$ .

**Example 11** Suppose that

$$H(z) = \frac{4z^3 - 10z^2 - z - 3}{4z^3 - 4z^2 + z - 1}$$

First substitute  $z = \frac{1}{\zeta}$  and simplify by multiplying both the numerator and denominator by  $-\zeta^3$ .

$$H\left(\frac{1}{\zeta}\right) = \frac{\frac{4}{\zeta^3} - \frac{10}{\zeta^2} - \frac{1}{\zeta} - 3}{\frac{4}{\zeta^3} - \frac{4}{\zeta^2} + \frac{1}{\zeta} - 1} = \frac{3\zeta^3 + \zeta^2 + 10\zeta - 4}{\zeta^3 - \zeta^2 + 4\zeta - 4}$$

Both the numerator and denominator have degree 3. Because they have the same degree, we can avoid doing long division (for a review of long division of polynomials, see the notes “Long Division”) by the simple expedient of, first, observing that the coefficient of  $\zeta^3$  in the numerator is 3 times the coefficient of  $\zeta^3$  in the denominator and, second, adding and subtracting 3 times the denominator to the numerator:

$$H\left(\frac{1}{\zeta}\right) = \frac{\text{numerator}}{\text{denominator}} = \frac{3 \text{ denominator} + \text{numerator} - 3 \text{ denominator}}{\text{denominator}} = 3 + \frac{\text{numerator} - 3 \text{ denominator}}{\text{denominator}}$$

The point here is that in

$$\text{numerator} - 3 \text{ denominator} = 3\zeta^3 + \zeta^2 + 10\zeta - 4 - 3(\zeta^3 - \zeta^2 + 4\zeta - 4) = 4\zeta^2 - 2\zeta + 8$$

the degree three terms, which are the highest degree terms, have cancelled out. So

$$H\left(\frac{1}{\zeta}\right) = 3 + \frac{4\zeta^2 - 2\zeta + 8}{\zeta^3 - \zeta^2 + 4\zeta - 4}$$

Next, we have to factor the denominator  $D(\zeta) = \zeta^3 - \zeta^2 + 4\zeta - 4$ . There is a MATLAB command, named `roots`, that will find (approximately, but accurately) the roots of any polynomial. But this is a rigged up example with integer roots. There is root finding trick that works well on exams and problem sets, where integer roots are common. It is simply the fact that any integer root of a polynomial with integer coefficients must divide the constant term, in this case  $-4$ , exactly. So the only possible integer roots are  $\pm 1, \pm 2$  and  $\pm 4$ . We just substitute these candidates, starting with the simplest, namely 1, into the denominator. Since  $D(1) = 0$ , 1 is a root and  $\zeta - 1$  must divide  $D(\zeta)$  exactly. So  $D(\zeta)$  must be  $(\zeta - 1)$  times a quadratic polynomial:

$$D(\zeta) = \zeta^3 - \zeta^2 + 4\zeta - 4 = (\zeta - 1)(a\zeta^2 + b\zeta + c)$$

The coefficients  $a$ ,  $b$  and  $c$  can be found by long division. But in this case, it is easier to observe that the product  $(\zeta - 1)(a\zeta^2 + b\zeta + c) = a\zeta^3 + (b - a)\zeta^2 + (c - b)\zeta - c$  is  $\zeta^3 - \zeta^2 + 4\zeta - 4$  if and only if  $a = 1$ ,  $c = 4$  and  $b = 0$ . Hence  $D(\zeta) = (\zeta - 1)(\zeta^2 + 4) = (\zeta - 1)(\zeta + 2i)(\zeta - 2i)$  and

$$H\left(\frac{1}{\zeta}\right) = 3 + \frac{a_1}{\zeta - 1} + \frac{a_2}{\zeta + 2i} + \frac{a_3}{\zeta - 2i}$$

with the coefficients  $a_i$  determined by

$$\frac{a_1}{\zeta - 1} + \frac{a_2}{\zeta + 2i} + \frac{a_3}{\zeta - 2i} = \frac{4\zeta^2 - 2\zeta + 8}{\zeta^3 - \zeta^2 + 4\zeta - 4} = \frac{R(\zeta)}{D(\zeta)}$$

Since  $D'(\zeta) = 3\zeta^2 - 2\zeta + 4$

$$\begin{aligned} a_1 &= \frac{R(1)}{D'(1)} = \frac{4 - 2 + 8}{3 - 2 + 4} = 2 \\ a_2 &= \frac{R(-2i)}{D'(-2i)} = \frac{-16 + 4i + 8}{-12 + 4i + 4} = 1 \\ a_3 &= \frac{R(2i)}{D'(2i)} = \frac{-16 - 4i + 8}{-12 - 4i + 4} = 1 \end{aligned}$$

and

$$H\left(\frac{1}{\zeta}\right) = 3 + \frac{2}{\zeta - 1} + \frac{1}{\zeta + 2i} + \frac{1}{\zeta - 2i} = 3 - 2\frac{1}{1 - \zeta} + \frac{1}{2i} \frac{1}{1 + \zeta/2i} - \frac{1}{2i} \frac{1}{1 - \zeta/2i}$$

From the table

$$\begin{aligned}
 h[n] &= 3\delta[n] - 2e_1[n] + \frac{1}{2i}e_{-1/2i}[n] - \frac{1}{2i}e_{1/2i}[n] \\
 &= \left\{ \begin{array}{ll} 0 & \text{if } n < 0 \\ 3 - 2 + \frac{1}{2i} - \frac{1}{2i} & \text{if } n = 0 \\ -2(1)^n + \frac{1}{2i}\left(-\frac{1}{2i}\right)^n - \frac{1}{2i}\left(\frac{1}{2i}\right)^n & \text{if } n > 0 \end{array} \right\} = \left\{ \begin{array}{ll} 0 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ -2 & \text{if } n > 0, n \text{ even} \\ -2 - \frac{2}{2i}\left(\frac{1}{2i}\right)^n & \text{if } n > 0, n \text{ odd} \end{array} \right\} \\
 &= \left\{ \begin{array}{ll} 0 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ -2 & \text{if } n > 0, n \text{ even} \\ -2 - (-i)^{n+1}\frac{1}{2^n} & \text{if } n > 0, n \text{ odd} \end{array} \right\} = \left\{ \begin{array}{ll} 0 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ -2 & \text{if } n > 0, n \text{ even} \\ -2 - (-1)^{\frac{n+1}{2}}\frac{1}{2^n} & \text{if } n > 0, n \text{ odd} \end{array} \right\}
 \end{aligned}$$

One can avoid substituting  $\zeta = \frac{1}{z}$  when recovering a causal impulse response function from its (rational)  $z$ -transform by the simple expedient of adding  $\frac{1}{z-a}$  to the lookup table. We already know that  $a^n u[n]$  has  $z$ -transform  $\frac{1}{1-a/z} = \frac{z}{z-a}$ . So by the “Time Shifting” property (see the table of properties below)  $a^{n-1}u[n-1]$  has  $z$ -transform  $z^{-1}\frac{z}{z-a} = \frac{1}{z-a}$  and we have the enlarged lookup table

$z$ -Transform	Signal
$\frac{1}{z^m}$	$\delta_m[n] = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise} \end{cases}$
$\frac{1}{1-a/z}$	$e_a[n] = a^n u[n] = \begin{cases} a^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$
$\frac{1}{z-a}$	$e_a[n-1] = a^{n-1}u[n-1] = \begin{cases} a^{n-1} & \text{if } n \geq 1 \\ 0 & \text{if } n < 1 \end{cases}$

### Properties of the $z$ -transform

It is not surprising that the  $z$ -transform shares many properties with Fourier series and Fourier transforms. Here is a table of a few of the more important ones.

Property	Aperiodic Signal	$z$ -Transform	Region of Convergence
	$x[n]$	$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$	$R_X$
	$y[n]$	$Y(z) = \sum_{n=-\infty}^{\infty} y[n]Z^{-n}$	$R_Y$
Linearity	$Ax[n] + By[n]$	$AX(z) + BY(z)$	at least $R_X \cap R_Y$
Time Shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	at least $R_X \cap \{ z  > 0\}$
$z$ -scaling	$a^n x[n]$	$X\left(\frac{z}{a}\right)$	$\{az \mid z \text{ in } R_X\}$
Time Reversal	$x[-n]$	$X\left(\frac{1}{z}\right)$	$\left\{\frac{1}{z} \mid z \text{ in } R_X\right\}$
Conjugation	$\overline{x[n]}$	$\overline{X(\overline{z})}$	$R_X$
$z$ -Differentiation	$nx[n]$	$-z\frac{dX}{dz}(z)$	at least the interior of $R_X$
Convolution	$\sum_{m=-\infty}^{\infty} x[m]y[n-m]$	$X(z)Y(z)$	at least $R_X \cap R_Y$

In the  $z$ -Differentiation property the “interior of  $R_X$ ” means all of  $R_X$  except for its boundary. For example, if  $R_X = \{a \leq |z| \leq b\}$ , then the interior of  $R_X$  is  $\{a < |z| < b\}$ . If  $R_X = \{a \leq |z| < b\}$  or  $R_X = \{a < |z| < b\}$ ,

then the interior of  $R_X$  is still  $\{a < |z| < b\}$ . The notation  $A \cap B$  which appears in the ROC column of several properties is read “the intersection of  $A$  and  $B$ ”. It consists of all  $z$ ’s that are in both  $A$  and  $B$ .

The derivations of the linearity, time shifting,  $z$ -scaling, time reversal, conjugation, difference and  $z$ -differentiation properties are trivial computations. Here is the derivation of the convolution property. Denote  $h[n] = \sum_{m=-\infty}^{\infty} x[m]y[n-m]$ . Then

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \sum_{m,n} x[m]y[n-m]z^{-n} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[m]z^{-m} y[n-m]z^{-(n-m)}$$

Now make a change of summation variables with  $n$  being replaced by  $k = n - m$ . Note that, for each fixed  $m$ , as  $n$  runs over all integers from  $-\infty$  to  $+\infty$ ,  $k$  also runs over all integers from  $-\infty$  to  $+\infty$ . So

$$H(z) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[m]z^{-m} y[k]z^{-k} = X(z)Y(z)$$

as desired.

**Example 12** We have already seen that, for the banking Example 2, the input and output signals are related by

$$y[n] = ay[n-1] + x[n] \quad (8)$$

with  $a = 1 + \frac{r}{100}$ . We have also already computed the impulse response function for this system in Example 4. But here is another computation of the impulse response function that uses the time shifting property of the  $z$ -transform. Let’s denote by  $Y(z)$ , the  $z$ -transform of  $y[n]$  and by  $X(z)$ , the  $z$ -transform of  $x[n]$ . By the time shifting property, the  $z$ -transform of  $y[n-1]$  is  $z^{-1}Y(z)$ . So, taking the  $z$ -transform of (8) gives

$$Y(z) = \frac{a}{z}Y(z) + X(z) \implies Y(z) = \frac{1}{1 - \frac{a}{z}}X(z) \quad (9)$$

When  $x[n] = \delta[n]$ , the  $z$ -transform of  $x[n]$  is  $X(z) = 1$ . In this case, we have that  $Y(z) = \frac{1}{1 - \frac{a}{z}}$ . This is, of course, exactly the  $z$ -transform of the impulse response function  $h[n] = a^n u[n]$  that we found in Example 4.

**Example 13** Now, as a continuation of the last example, suppose that we start the account with an initial deposit of  $D$  in month 0. Then, starting in month 1, we make withdrawals. Say we withdraw  $W$  in month 1,  $Wb$  in month 2,  $Wb^2$  in month 3 and so on. Then

$$x[n] = \begin{cases} D & \text{if } n = 0 \\ -W & \text{if } n = 1 \\ -Wb & \text{if } n = 2 \\ \vdots & \end{cases} = D\delta[n] - Wb^{n-1}u[n-1]$$

Since

- the  $z$ -transform of  $\delta[n]$  is 1,
- the  $z$ -transform of  $b^n u[n]$  is  $\frac{1}{1-b/z}$ , and
- the  $z$ -transform of  $b^{n-1}u[n-1]$  is  $\frac{1}{z} \frac{1}{1-b/z}$ , by the time shift property,

we have

$$X(z) = D - W \frac{1}{z} \frac{1}{1-b/z}$$

and

$$Y(z) = \frac{1}{1 - \frac{a}{z}} X(z) = \frac{1}{1 - \frac{a}{z}} \left[ D - W \frac{1}{z} \frac{1}{1 - \frac{b}{z}} \right] = \frac{D}{1 - \frac{a}{z}} - W \frac{1}{z} \frac{1}{1 - \frac{a}{z}} \frac{1}{1 - \frac{b}{z}}$$

The contribution of  $\frac{D}{1 - \frac{a}{z}}$  to  $y[n]$  is, of course,  $Da^n u[n]$ . To use partial fractions to determine the contribution of the other term, write  $\zeta = \frac{1}{z}$ . Then, the second term is  $-W$  times

$$\frac{\zeta}{(1 - a\zeta)(1 - b\zeta)} = \frac{A}{1 - a\zeta} + \frac{B}{1 - b\zeta}$$

with

$$A = \left. \frac{\zeta}{(1 - b\zeta)} \right|_{a\zeta=1} = \frac{\frac{1}{a}}{1 - \frac{b}{a}} = \frac{1}{a - b} \quad B = \left. \frac{\zeta}{(1 - a\zeta)} \right|_{b\zeta=1} = \frac{\frac{1}{b}}{1 - \frac{a}{b}} = \frac{1}{b - a}$$

(assuming that  $b \neq a$ ). Thus

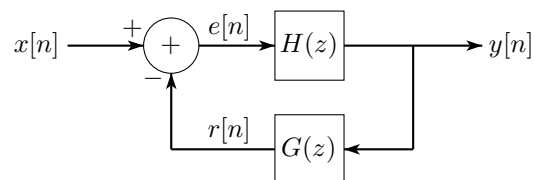
$$Y(z) = D \frac{1}{1 - \frac{a}{z}} - \frac{W}{a - b} \frac{1}{1 - \frac{a}{z}} - \frac{W}{b - a} \frac{1}{1 - \frac{b}{z}}$$

and

$$y[n] = \left( D - \frac{W}{a-b} \right) a^n u[n] + \frac{W}{a-b} b^n u[n]$$

## Feedback systems

Automatic control systems typically use the output of the system to modify the input. This is called feedback. A typical example is a household heating/cooling system with thermostatic control. The householder sets the thermostat at a desired temperature. That's the input signal. A thermometer in the control reads the current room temperature, which is the output signal. If the room temperature is too low the furnace is turned on. If the temperature is too high, the air-conditioner is turned on. When the desired temperature is reached, the furnace/air-conditioner is turned back off. A general discrete-time, linear, feedback system has the form



Here

- $x[n]$  is the input signal (for example the desired temperature set by the householder) at time  $n$ .
- $H(z)$  is called *the system function of the forward path*. In our example, it is the  $z$ -transform of the impulse response that determines the behaviour of the furnace/air-conditioner.
- $y[n]$  is the output signal (for example the actual room temperature) at time  $n$ .
- $y[n]$  also acts as the input signal to the subsystem with  $z$ -transformed impulse response  $G(z)$ , which is called *the system function of the feedback path*. In our example, it is the impulse response that determines the behaviour of the controller in the thermostat.
- The circular *adder* in the block diagram above combines the user input signal  $x[n]$  and the output  $r[n]$  of the controller to form the *error signal*  $e[n] = x[n] - r[n]$  (for example the difference between the desired temperature and the measured temperature), which is the input signal fed to the  $H$  subsystem.

Suppose that

$$H(z) = \sum_{k=-\infty}^{\infty} z^{-k} h[k] \quad G(z) = \sum_{k=-\infty}^{\infty} z^{-k} g[k]$$

Since the input to the  $H$  subsystem is  $e[n]$  and the input to the  $G$  subsystem is  $y[n]$

$$y[n] = (e * h)[n] \quad r[n] = (y * g)[n]$$

Taking the  $z$ -transform of both of these equations gives

$$Y(z) = E(z)H(z) \quad R(z) = Y(z)G(z)$$

Since  $e[n] = x[n] - r[n]$

$$E(z) = X(z) - R(z)$$

so that

$$Y(z) = (X(z) - R(z))H(z) = (X(z) - Y(z)G(z))H(z) = X(z)H(z) - Y(z)G(z)H(z)$$

Solving for  $Y(z)$ ,

$$Y(z) = \frac{H(z)}{1+G(z)H(z)}X(z) = Q(z)X(z) \text{ with } Q(z) = \frac{H(z)}{1+G(z)H(z)}$$

This tells us the  $z$  transform,  $Q(z)$ , of the impulse response of the full feedback system, including both the  $H$  and  $G$  subsystems.

**Example 14 (stabilization)** One important application of feedback is to stabilize otherwise unstable systems, like aircraft and rockets. Suppose for example, that

$$H(z) = \frac{b}{1-a/z} = b\frac{z}{z-a}$$

with region of convergence  $|z| > |a|$ , similar to Examples 7 and 10. If  $|a| > 1$  this system is causal but unstable. Also suppose that the feedback subsystem is just an amplifier, as in Example 5, so that

$$G(z) = K$$

Then the composite feedback system has

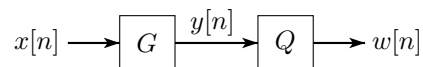
$$Q(z) = \frac{H(z)}{1+G(z)H(z)} = \frac{b\frac{z}{z-a}}{1+Kb\frac{z}{z-a}} = \frac{bz}{z-a+Kbz} = \frac{bz}{(1+Kb)z-a}$$

which has a single pole at  $z = \frac{a}{1+Kb}$ . If the amplifier is strong enough that  $|1+Kb| > |a|$ , the pole has modulus  $|\frac{a}{1+Kb}| < 1$  and the feedback system is stable.

**Example 15 (inversion)** If we choose the  $H$  subsystem to be an amplifier with  $H(z) = K$ , then

$$Q(z) = \frac{K}{1+G(z)K} \approx \frac{1}{G(z)}$$

provided  $|KG(z)| \gg 1$ . That is, provided  $K \gg |G(z)|^{-1}$ . So we have constructed an approximate inverse for the system with system function  $G(z)$ . Here is what that means. Suppose we cascade the  $G$  system and the  $Q$  system, so that the output of the  $G$  system is fed into the input of the  $Q$  system as in the figure



Then the output,  $w[n]$ , of the cascaded system is

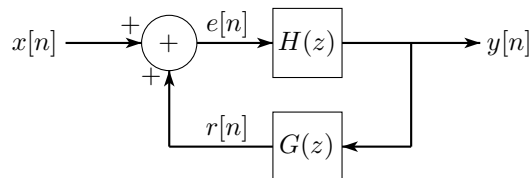
$$w[n] = (q * y)[n] \quad \text{with} \quad y[n] = (g * x)[n]$$

Taking  $z$  transforms,

$$W(z) = Q(z)Y(z) \quad \text{with} \quad Y(z) = G(z)X(z) \quad \implies \quad W(z) = Q(z)G(z)X(z)$$

If  $Q(z)$  were exactly  $\frac{1}{G(z)}$ , then we would have  $W(z)$  exactly the same as  $X(z)$ , so that the final output  $w[n]$  would be identical to the initial input  $x[n]$ .

**Example 16 (positive feedback)** Suppose that our feedback system is modified so that the output,  $r[n]$ , of the feedback path is added, rather than subtracted, from the input  $x[n]$ . This is called *positive feedback*. Effectively, we have replaced  $G(z)$  by  $-G(z)$ . So the feedback system now has system function



$$Q(z) = \frac{H(z)}{1 - G(z)H(z)}$$

Now if, as in Example 14,  $H(z) = \frac{b}{1-a/z} = b\frac{z}{z-a}$  and  $G(z) = K$ , then

$$Q(z) = \frac{H(z)}{1 - G(z)H(z)} = \frac{b\frac{z}{z-a}}{1 - Kb\frac{z}{z-a}} = \frac{bz}{z-a-Kbz} = \frac{bz}{(1-Kb)z-a}$$

which has a single pole at  $z = \frac{a}{1-Kb}$ . Even if the original system had  $|a| < 1$ , and so was stable, the feedback system becomes unstable when  $K$  is large enough that  $Kb$  is close enough to 1 that  $|\frac{a}{1-Kb}| > 1$ . Generation of instability through positive feedback is commonly observed in squealing loudspeaker systems.