Long Division

By definition, a rational function is the ratio of two polynomials $\frac{N(x)}{D(x)}$. In the event that the degree of the numerator, N, is greater than or equal to the degree of the denominator, D, long division may be used to rewrite $\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$ with P and R polynomials and with the degree of the R strictly less than the degree of D. Here are two examples.

Example 1 The rational function

$$\frac{x^4 + 5x^3 + 16x^2 + 26x + 22}{x^3 + 3x^2 + 7x + 5}$$

has a numerator of degree 4 and denominator of degree 3. So we long divide, just like in public school. The term of highest power in the denominator, x^3, divides into the term of highest power in the numerator, x^4, to give x. So we put x above the division symbol and subtract x times the denominator from the numerator. This gives a remainder of $2x^3 + 9x^2 + 21x + 22$. The term of highest power in the denominator, x^3, divides into the term of highest power in the remainder, $2x^3$, to give 2. So we add a 2 above the division symbol and subtract 2 times the denominator from the remainder. This gives a new remainder of $3x^2 + 7x + 12$.

$$\begin{array}{r|l l}
 & x + \quad 2 & \\
\cline{2-2}
x^3 + 3x^2 + 7x + 5 & x^4 + 5x^3 + 16x^2 + 26x + 22 & \\
 & x^4 + 3x^3 + \ \ 7x^2 + \ \ 5x & \longleftarrow x(x^3 + 3x^2 + 7x + 5) \\
\cline{2-2}
 & \qquad 2x^3 + \ \ 9x^2 + 21x + 22 & \\
 & \qquad 2x^3 + \ \ 6x^2 + 14x + 10 & \longleftarrow 2(x^3 + 3x^2 + 7x + 5) \\
\cline{2-2}
 & \qquad\qquad 3x^2 + \ \ 7x + 12 &
\end{array}$$

In this example, when we subtracted $x(x^3+3x^2+7x+5)$ and $2(x^3+3x^2+7x+5)$ from $x^4+5x^3+16x^2+26x+22$ we ended up with $3x^2 + 7x + 12$. That is,

$$x^4 + 5x^3 + 16x^2 + 26x + 22 \ - x(x^3 + 3x^2 + 7x + 5) \ - 2(x^3 + 3x^2 + 7x + 5) \ = \ 3x^2 + 7x + 12$$

so that

$$x^4 + 5x^3 + 16x^2 + 26x + 22 \ = \ (x + 2)(x^3 + 3x^2 + 7x + 5) \ + \ 3x^3 + 7x + 12$$

Dividing across by $(x^3 + 3x^2 + 7x + 5)$, this tells us that

$$\boxed{\frac{x^4 + 5x^3 + 16x^2 + 26x + 22}{x^3 + 3x^2 + 7x + 5} = x + 2 + \frac{3x^3 + 7x + 12}{x^3 + 3x^2 + 7x + 5}}$$

Example 2 The rational function

$$\frac{x^3 + 3x^2 + 7x + 5}{x + 1}$$

has a numerator of degree 3 and denominator of degree 1. Again, we long divide. The term of highest power in the denominator, x, divides into the term of highest power in the numerator, x^3, to give x^2. So we put x^2 above the division symbol and subtract x^2 times the denominator from the numerator. This gives a remainder of $2x^2 + 7x + 5$. The term of highest power in the denominator, x, divides into the term of highest power in the remainder, $2x^2$, to give $2x$. So we add a $2x$ above the division symbol and subtract

$2x$ times the denominator from the remainder. This gives a new remainder of $5x + 5$. The term of highest power in the denominator, x, divides into the term of highest power in the remainder, $5x$, to give 5. So we add a 5 above the division symbol and subtract 5 times the denominator from the remainder. This gives a new remainder of 0.

$$
\begin{array}{r|l l}
 & x^2 + 2x + 5 & \\
\hline
x+1 & x^3 + 3x^2 + 7x + 5 & \\
 & x^3 + x^2 & \longleftarrow x^2(x+1) \\
\hline
 & \quad 2x^2 + 7x + 5 & \\
 & \quad 2x^2 + 2x & \longleftarrow 2x(x+1) \\
\hline
 & \qquad 5x + 5 & \\
 & \qquad 5x + 5 & \longleftarrow 5(x+1) \\
\hline
 & \qquad\quad 0 &
\end{array}
$$

In this example, when we subtracted $x^2(x+1)$ and $2x(x+1)$ and $5(x+1)$ from $x^3 + 3x^2 + 7x + 5$ we ended up with 0. Hence

$$x^3 + 3x^2 + 7x + 5 - x^2(x+1) - 2x(x+1) - 5(x+1) = 0$$

or

$$x^3 + 3x^2 + 7x + 5 = x^2(x+1) + 2x(x+1) + 5(x+1)$$

Dividing across by $x + 1$ gives

$$\boxed{\frac{x^3 + 3x^2 + 7x + 5}{x+1} = x^2 + 2x + 5}$$

exactly.