Solution of the Heat Equation by Separation of Variables

The Problem

Let $u(x,t)$ denote the temperature at position x and time t in a long, thin rod of length ℓ that runs from $x=0$ to $x=\ell$. Assume that the sides of the rod are insulated so that heat energy neither enters nor leaves the rod through its sides. Also assume that heat energy is neither created nor destroyed (for example by chemical reactions) in the interior of the rod. Then $u(x,t)$ obeys the heat equation

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) \qquad \text{for all } 0 < x < \ell \text{ and } t > 0 \tag{1}$$

This equation was derived in the notes "The Heat Equation (One Space Dimension)".

Suppose further that the temperature at the ends of the rod is held fixed at 0. This information is encoded in the "boundary conditions"

$$u(0,t) = 0 \qquad \text{for all } t > 0 \tag{2}$$

$$u(\ell,t) = 0 \qquad \text{for all } t > 0 \tag{3}$$

Finally, also assume that we know the temperature throughout the rod time 0. So there is some given function $f(x)$ such that the "initial condition"

$$u(x,0) = f(x) \qquad \text{for all } 0 < x < \ell \tag{4}$$

is satisfied. The problem is to determine $u(x,t)$ for all x and t.

Outline of the Method of Separation of Variables

We are going to solve this problem using the same three steps that we used in solving the wave equation.

Step 1 In the first step, we find all solutions of (1) that are of the special form $u(x,t) = X(x)T(t)$ for some function $X(x)$ that depends on x but not t and some function $T(t)$ that depends on t but not x. Once again, if we find a bunch of solutions $X_i(x)T_i(t)$ of this form, then since (1) is a linear equation, $\sum_i a_i X_i(x)T_i(t)$ is also a solution for any choice of the constants a_i.

Step 2 We impose the boundary conditions (2) and (3).

Step 3 We impose the initial condition (4).

The First Step – Finding Factorized Solutions

The factorized function $u(x,t) = X(x)T(t)$ is a solution to the heat equation (1) if and only if

$$X(x)T'(t) = \alpha^2 X''(x)T(t) \iff \frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)}$$

The left hand side is independent of t. The right hand side is independent of x. The two sides are equal. So both sides must be independent of both x and t and hence equal to some constant, say σ. So we have

$$\begin{aligned} &\frac{X''(x)}{X(x)} = \sigma & \qquad & \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \sigma \\ \iff \qquad & X''(x) - \sigma X(x) = 0 & & T'(t) - \alpha^2 \sigma T(t) = 0 \end{aligned} \tag{5}$$

If $\sigma \neq 0$, the general solution to (5) is

$$X(x) = d_1 e^{\sqrt{\sigma}x} + d_2 e^{-\sqrt{\sigma}x} \qquad T(t) = d_3 e^{\alpha^2 \sigma t}$$

for arbitrary constants d_1, d_2 and d_3. If $\sigma = 0$, the equations (5) simplify to

$$X''(x) = 0 \qquad T'(t) = 0$$

and the general solution is

$$X(x) = d_1 + d_2 x \qquad T(t) = d_3$$

for arbitrary constants d_1, d_2 and d_3. We have now found a huge number of solutions to the heat equation (1). Namely

$$\begin{aligned} u(x,t) &= \big(d_1 e^{\sqrt{\sigma}x} + d_2 e^{-\sqrt{\sigma}x}\big)\big(d_3 e^{\alpha^2 \sigma t}\big) && \text{for arbitrary } \sigma \neq 0 \text{ and arbitrary } d_1, d_2, d_3 \\ u(x,t) &= \big(d_1 + d_2 x\big) d_3 && \text{for arbitrary } d_1, d_2, d_3 \end{aligned}$$

The Second Step – Imposition of the Boundary Conditions

If $X_i(x)T_i(t)$, $i = 1, 2, 3, \cdots$ all solve the heat equation (1), then $\sum_i a_i X_i(x) T_i(t)$ is also a solution for any choice of the constants a_i. This solution satisfies the boundary condition (2) if and only if

$$\sum_i a_i X_i(0) T_i(t) = 0 \qquad \text{for all } t > 0$$

This will certainly be the case if $X_i(0) = 0$ for all i. Similarly, $u(x,t) = \sum_i a_i X_i(x) T_i(t)$ satisfies the boundary condition (3) if and only if

$$\sum_i a_i X_i(\ell) T_i(t) = 0 \qquad \text{for all } t > 0$$

and this will certainly be the case if $X_i(\ell) = 0$ for all i. We are now going to go through the solutions that we found in Step 1 and discard all of those that fail to satisfy $X(0) = X(\ell) = 0$.

First, consider $\sigma = 0$ so that $X(x) = d_1 + d_2 x$. The condition $X(0) = 0$ is satisfied if and only if $d_1 = 0$. The condition $X(\ell) = 0$ is satisfied if and only if $d_1 + \ell d_2 = 0$. So the conditions $X(0) = X(\ell) = 0$ are both satisfied only if $d_1 = d_2 = 0$, in which case $X(x)$ is identically zero. There is nothing to be gained by keeping an identically zero $X(x)$, so we discard $\sigma = 0$ completely.

Next, consider $\sigma \neq 0$ so that $d_1 e^{\sqrt{\sigma}x} + d_2 e^{-\sqrt{\sigma}x}$. The condition $X(0) = 0$ is satisfied if and only if $d_1 + d_2 = 0$. So we require that $d_2 = -d_1$. The condition $X(\ell) = 0$ is satisfied if and only if

$$0 = d_1 e^{\sqrt{\sigma}\ell} + d_2 e^{-\sqrt{\sigma}\ell} = d_1\big(e^{\sqrt{\sigma}\ell} - e^{-\sqrt{\sigma}\ell}\big)$$

If d_1 were zero, then $X(x)$ would again be identically zero and hence useless. So instead, we discard any σ that does not obey

$$\begin{aligned} e^{\sqrt{\sigma}\ell} - e^{-\sqrt{\sigma}\ell} = 0 &\iff e^{\sqrt{\sigma}\ell} = e^{-\sqrt{\sigma}\ell} \iff e^{2\sqrt{\sigma}\ell} = 1 \iff 2\sqrt{\sigma}\ell = 2k\pi\imath \iff \sqrt{\sigma} = k\tfrac{\pi}{\ell}\imath \\ &\iff \sigma = -k^2\tfrac{\pi^2}{\ell^2} \end{aligned}$$

for some integer k. With $\sqrt{\sigma} = k\frac{\pi}{\ell}\imath$ and $d_2 = -d_1$,

$$\begin{aligned} X(x)T(t) &= d_1\big(e^{\imath\frac{k\pi}{\ell}x} - e^{-\imath\frac{k\pi}{\ell}x}\big)\big(d_3 e^{-\alpha^2\pi^2k^2t/\ell^2}\big) = 2\imath d_1 d_3 \sin\big(\tfrac{k\pi}{\ell}x\big) e^{-\alpha^2\pi^2k^2t/\ell^2} \\ &= \beta_k \sin\big(\tfrac{k\pi}{\ell}x\big) e^{-\alpha^2\pi^2k^2t/\ell^2} \end{aligned}$$

where $\beta_k = -2id_1d_3$.

The Third Step – Imposition of the Initial Condition

We now know that

$$u(x,t) = \sum_{k=1}^{\infty} \beta_k \sin\left(\tfrac{k\pi}{\ell}x\right) e^{-\alpha^2\pi^2k^2t/\ell^2}$$

obeys the heat equation (1) and the boundary conditions (2) and (3), for any choice of the constants β_k. It remains only to see if we can choose the β_k's to satisfy

$$f(x) = u(x,0) = \sum_{k=1}^{\infty} \beta_k \sin\left(\tfrac{k\pi}{\ell}x\right) \tag{4'}$$

But any (reasonably smooth) function, $f(x)$, defined on the interval $0 < x < \ell$, has a unique representation[(1)] of the form (4′) and the coefficients in this representation are given by

$$\beta_k = \tfrac{2}{\ell}\int_0^\ell f(x) \sin\tfrac{k\pi x}{\ell}\ dx$$

So we have a solution:

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} \beta_k \sin\left(\tfrac{k\pi}{\ell}x\right) e^{-\alpha^2\pi^2k^2t/\ell^2}} \qquad \text{with} \qquad \boxed{\beta_k = \tfrac{2}{\ell}\int_0^\ell f(x) \sin\tfrac{k\pi x}{\ell}\ dx} \tag{6}$$

The behaviour of the solution (6) is very different from the corresponding solution of the wave equation. Each exponential $e^{-\alpha^2\pi^2k^2t/\ell^2}$ converges to zero as $t \to \infty$. Hence $u(x,t)$ tends, very reasonably, to the equilibrium temperature zero.

Inhomogeneous Boundary Conditions

Consider the following problem. The temperature, $u(x,t)$, in a metal rod of unit length satisfies

$$u_t = 9u_{xx}, \qquad 0 \le x \le 2,\ t \ge 0.$$

The ends of the rod at $x = 0$ and $x = 2$, are maintained at a constant temperatures of 0 and 8 respectively, so that the boundary conditions are

$$u(0,t) = 0, \qquad u(2,t) = 8, \qquad t \ge 0$$

The initial temperature distribution is $u(x,0) = 2x^2$. Find the temperature for $t \ge 0$.

Warning. With the exception of the boundary condition $u(2,t) = 8$, this problem is a special case of the problem (1–4) with $\alpha = 3$, $\ell = 2$ and $f(x) = 2t^2$. But the boundary condition $u(2,t) = 8$ cannot be achieved by requiring $X(2) = 8$. The reason is that "$X_i(2) = 8$ for all i" does not imply that "$u(2,t) = \sum_i X_i(2)T_i(t) = 8$ for all t. Fortunately, with a little physical intuition, we can reduce the current problem to (1–4).

Motivation. We would expect that as $t \to \infty$ the temperature approaches an equilibrium temperature that increases from 0 at $x = 0$ to 8 at $x = 2$. The simplest possible equilibrium temperature is $v(x) = 4x$. As supporting evidence for this guess, note that $u(x,t) = 4x$ is a steady state (i.e. time independent)

(1) See, for example, the notes "Fourier Series".

solution of the heat equation $u_t = 9u_{xx}$ which also satisfies the boundary conditions. So the transient[1] $w(x,t) = u(x,t) - v(x)$ obeys the boundary conditions

$$w(0,t) = u(0,t) - v(0) = 0 - 0 = 0 \qquad w(2,t) = u(2,t) - v(2) = 8 - 8 = 0$$

which we already know how to handle.

Solution. We have not yet verified that $u(x,t)$ tends to $4x$ as $t \to \infty$. We may non–the–less define $w(x,t) = u(x,t) - v(x) = u(x,t) - 4x$. If our intuition is correct, this will be the transient part of the solution, decaying to zero for large t. We first find $w(x,t)$.

Subtracting	$0 = v_t = 9v_{xx}$	from	$u_t = 9u_{xx}$	gives	$(u-v)_t = 9(u-v)_{xx}$ or $w_t = 9w_{xx}$.
Subtracting	$v(0) = 0$	from	$u(0,t) = 0$	gives	$w(0,t) = 0$.
Subtracting	$v(2) = 8$	from	$u(2,t) = 8$	gives	$w(2,t) = 0$.
Subtracting	$v(x) = 4x$	from	$u(x,0) = 2x^2$	gives	$w(x,0) = 2x^2 - 4x$.

Thus $u(x,t)$ is a solution for the original problem if and only if $w(x,t)$ obeys

$$w_t = 9w_{xx}, \qquad w(0,t) = 0, \qquad w(2,t) = 0, \qquad w(x,0) = 2x^2 - 4x, \qquad 0 \le x \le 2,\ t \ge 0 \tag{$*$}$$

This is now exactly (1–4) with u replaced by w and $\ell = 2$, $\alpha = 3$ and $f(x) = 2x^2 - 4x$. So, by (6),

$$w(x,t) = \sum_{k=1}^{\infty} \beta_k e^{-\frac{9}{4}\pi^2 k^2 t} \sin \tfrac{k\pi x}{2}$$

with

$$\begin{aligned}
\beta_k &= \tfrac{2}{\ell} \int_0^\ell f(x) \sin \tfrac{k\pi x}{\ell}\ dx = \int_0^2 (2x^2 - 4x) \sin \tfrac{k\pi x}{2}\ dx \\
&= \Big[\tfrac{32}{k^3\pi^3} \cos \tfrac{k\pi x}{2} + \tfrac{16x}{k^2\pi^2} \sin \tfrac{k\pi x}{2} - \tfrac{4x^2}{k\pi} \cos \tfrac{k\pi x}{2} - \tfrac{16}{k^2\pi^2} \sin \tfrac{k\pi x}{2} + \tfrac{8x}{k\pi} \cos \tfrac{k\pi x}{2} \Big]_0^2 \\
&= \Big[\tfrac{32}{k^3\pi^3} \cos \tfrac{k\pi x}{2} - \tfrac{4x^2}{k\pi} \cos \tfrac{k\pi x}{2} + \tfrac{8x}{k\pi} \cos \tfrac{k\pi x}{2} \Big]_0^2 \\
&= (-1)^k \Big[\tfrac{32}{k^3\pi^3} - \tfrac{16}{k\pi} + \tfrac{16}{k\pi} \Big] - \tfrac{32}{k^3\pi^3}
\end{aligned}$$

So the final answer is

$$u(x,t) = 4x + w(x,t) = 4x - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \tfrac{64}{k^3\pi^3} e^{-\frac{9}{4}\pi^2 k^2 t} \sin \tfrac{k\pi x}{2}$$

[1] That is, the part of the solution $u(x,t)$ which tends to zero as $t \to \infty$.