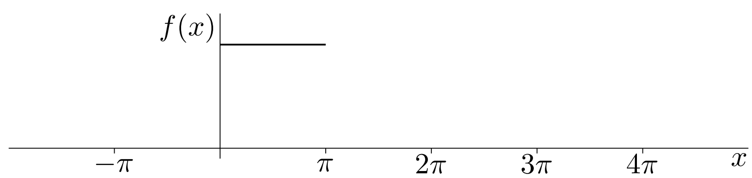
# Periodic Extensions

We know that every (sufficiently smooth) periodic function has a Fourier series expansion. It is fairly common for functions arising from certain applications to be defined only on a finite interval $0 < x < \ell$. This is the case if, for example, $f(x)$ is the vertical displacement of a string from the $x$ axis at position $x$ and if the string only runs from $x = 0$ to $x = \ell$. For the application, we only care about $x$'s between 0 and $\ell$. But we are free to extend the definition of $f(x)$ to $x < 0$ and $x > \ell$, for computational purposes.

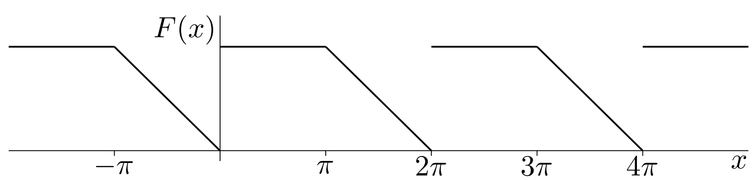
We define $F(x)$ to be a **periodic extension** of $f(x)$ if

$$i)\ F(x) = f(x) \quad \text{for} \quad 0 < x < \ell$$
$$ii)\ F(x) \text{ is periodic of period } 2\ell$$

There are many different periodic extensions of $f(x)$. Most of them are pretty useless. For example define $f(x) = 1$ for $0 < x < \pi$. We are leaving $f(x)$ undefined for $x \ge \pi$ or $x \le 0$.




Then the function $F(x)$ graphed in




is a perfectly legitimate, but virtually useless, periodic extension of $f$.

There are two periodic extensions for $f(x)$ that are very useful. Given any $f(x)$, which is defined only for $0 < x < \ell$, we define its **even periodic extension**, $F^e(x)$, by the conditions that

a) $F^e(x) = f(x)$ for $0 < x < \ell$
b) $F^e(x)$ is even (This fixes $F^e(x)$ for $-\ell < x < 0$.)
c) $F^e(x)$ has period $2\ell$ (This fixes $F^e(x)$ for all remaining $x$'s, except $x = n\pi$. )

and we define its **odd periodic extension**, $F^o(x)$, by

a) $F^o(x) = f(x)$ for $0 < x < \ell$
b) $F^o(x)$ is odd
c) $F^o(x)$ has period $2\ell$

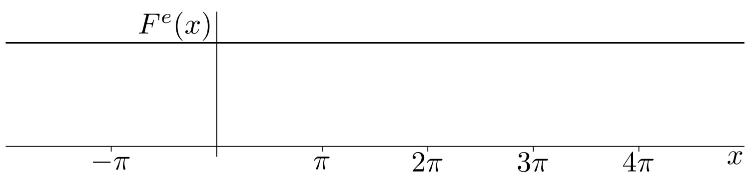
Because $F^e$ is even, it has a Fourier cosine expansion. Because $F^o$ is odd, it has a Fourier sine expansion. We have arranged that $f(x) = F^e(x) = F^o(x)$ for all $0 < x < \ell$. So, provided we restrict to $0 < x < \ell$, we have

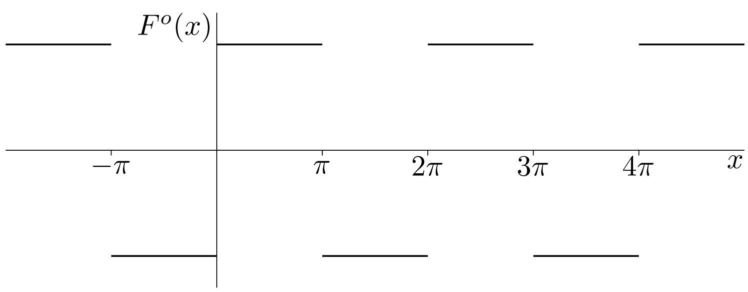
$$\begin{aligned} f(x) = F^e(x) &= \tfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\tfrac{k\pi x}{\ell}\right) \\ &= F^o(x) = \sum_{k=1}^{\infty} b_k \sin\left(\tfrac{k\pi x}{\ell}\right) \end{aligned}$$

where

$$\begin{aligned} a_k &= \tfrac{2}{\ell}\int_0^\ell F^e(x)\cos\left(\tfrac{k\pi x}{\ell}\right)dx = \tfrac{2}{\ell}\int_0^\ell f(x)\cos\left(\tfrac{k\pi x}{\ell}\right)dx \\ b_k &= \tfrac{2}{\ell}\int_0^\ell F^o(x)\sin\left(\tfrac{k\pi x}{\ell}\right)dx = \tfrac{2}{\ell}\int_0^\ell f(x)\sin\left(\tfrac{k\pi x}{\ell}\right)dx \end{aligned}$$

For example consider, again, the function $f(x)$ which is only defined for $0 < x < \pi$ and takes the value $f(x) = 1$ for all $0 < x < \pi$. The even and odd periodic extensions, $F^e(x)$ and $F^o(x)$ of this function are graphed below.




Both $F^e(x)$ and $F^o(x)$ take the value 1 for all $0 < x < \pi$. Both $F^e(x)$ and $F^o(x)$ have period $2\pi$. But $F^e(x)$ is an even function while $F^o(x)$ is an odd function. Because it is an odd periodic function with period $2\ell = 2\pi$, $F^o(x)$ has the Fourier series expansion

$$F^o(x) = \sum_{k=1}^{\infty} b_k \sin\left(\tfrac{k\pi x}{\ell}\right) = \sum_{k=1}^{\infty} b_k \sin(kx)$$

with

$$b_k = \tfrac{2}{\ell}\int_0^\ell F^o(x)\sin\left(\tfrac{k\pi x}{\ell}\right)\ dx = \tfrac{2}{\pi}\int_0^\pi 1\sin(kx)\ dx = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{4}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

That is

$$F^o(x) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \tfrac{4}{k\pi}\sin(kx)$$

Because it is an even periodic function with period $2\ell = 2\pi$, $F^e(x)$ has the Fourier series expansion

$$F^e(x) = \tfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\tfrac{k\pi x}{\ell}\right) = \tfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx)$$

with

$$a_k = \tfrac{2}{\ell}\int_0^{\ell} F^e(x)\cos\left(\tfrac{k\pi x}{\ell}\right)\ dx = \tfrac{2}{\pi}\int_0^{\pi} 1\cos(kx)\ dx = \begin{cases} 2 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

That is,

$$F^e(x) = \tfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) = 1 \qquad\qquad \text{(surprise!)}$$

For all $0 < x < \pi$ we have both

$$f(x) = F^e(x) = \tfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) = 1$$

$$f(x) = F^o(x) = \sum_{k=1}^{\infty} b_k \sin(kx) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \tfrac{4}{k\pi}\sin(kx)$$