Periodic Extensions

We know that every (sufficiently smooth) periodic function has a Fourier series expansion. It is fairly common for functions arising from certain applications to be defined only on a finite interval $0 < x < \ell$. This is the case if, for example, $f(x)$ is the vertical displacement of a string from the $x$ axis at position $x$ and if the string only runs from $x = 0$ to $x = \ell$. For the application, we only care about $x$’s between 0 and $\ell$. But we are free to extend the definition of $f(x)$ to $x < 0$ and $x > \ell$, for computational purposes.

We define $F(x)$ to be a periodic extension of $f(x)$ if

i) $F(x) = f(x)$ for $0 < x < \ell$

ii) $F(x)$ is periodic of period $2\ell$

There are many different periodic extensions of $f(x)$. Most of them are pretty useless. For example define $f(x) = 1$ for $0 < x < \pi$. We are leaving $f(x)$ undefined for $x \geq \pi$ or $x \leq 0$.

Then the function $F(x)$ graphed in

is a perfectly legitimate, but virtually useless, periodic extension of $f$.

There are two periodic extensions for $f(x)$ that are very useful. Given any $f(x)$, which is defined only for $0 < x < \ell$, we define its even periodic extension, $F^e(x)$, by the conditions that

a) $F^e(x) = f(x)$ for $0 < x < \ell$

b) $F^e(x)$ is even (This fixes $F^e(x)$ for $-\ell < x < 0$.)

c) $F^e(x)$ has period $2\ell$ (This fixes $F^e(x)$ for all remaining $x$’s, except $x = n\pi$. )

and we define its odd periodic extension, $F^o(x)$, by

a) $F^o(x) = f(x)$ for $0 < x < \ell$

b) $F^o(x)$ is odd

c) $F^o(x)$ has period $2\ell$

Because $F^e$ is even, it has a Fourier cosine expansion. Because $F^o$ is odd, it has a Fourier sine expansion. We have arranged that $f(x) = F^e(x) = F^o(x)$ for all $0 < x < \ell$. So, provided we
restrict to \(0 < x < \ell\), we have

\[
 f(x) = F^e(x) = \frac{x_0}{2} + \sum_{k=1}^{\infty} a_k \cos \left( \frac{k\pi x}{\ell} \right) \\
 = F^o(x) = \sum_{k=1}^{\infty} b_k \sin \left( \frac{k\pi x}{\ell} \right)
\]

where

\[
 a_k = \frac{2}{\ell} \int_0^\ell F^e(x) \cos \left( \frac{k\pi x}{\ell} \right) \, dx = \frac{2}{\ell} \int_0^\ell f(x) \cos \left( \frac{k\pi x}{\ell} \right) \, dx \\
b_k = \frac{2}{\ell} \int_0^\ell F^o(x) \sin \left( \frac{k\pi x}{\ell} \right) \, dx = \frac{2}{\ell} \int_0^\ell f(x) \sin \left( \frac{k\pi x}{\ell} \right) \, dx
\]

For example consider, again, the function \(f(x)\) which is only defined for \(0 < x < \pi\) and takes the value \(f(x) = 1\) for all \(0 < x < \pi\). The even and odd periodic extensions, \(F^e(x)\) and \(F^o(x)\) of this function are graphed below.

\[
 F^e(x) \\
 \begin{array}{cccccc}
 \, & -\pi & \pi & 2\pi & 3\pi & 4\pi & x \\
 \end{array}
\]

\[
 F^o(x) \\
 \begin{array}{cccccc}
 \, & -\pi & \pi & 2\pi & 3\pi & 4\pi & x \\
 \end{array}
\]

Both \(F^e(x)\) and \(F^o(x)\) take the value 1 for all \(0 < x < \pi\). Both \(F^e(x)\) and \(F^o(x)\) have period 2\(\pi\). But \(F^e(x)\) is an even function while \(F^o(x)\) is an odd function. Because it is an odd periodic function with period \(2\ell = 2\pi\), \(F^o(x)\) has the Fourier series expansion

\[
 F^o(x) = \sum_{k=1}^{\infty} b_k \sin \left( \frac{k\pi x}{\ell} \right) = \sum_{k=1}^{\infty} b_k \sin(kx)
\]

with

\[
 b_k = \frac{2}{\ell} \int_0^\ell F^o(x) \sin \left( \frac{k\pi x}{\ell} \right) \, dx = \frac{2}{\pi} \int_0^\pi 1 \sin(kx) \, dx = \begin{cases} 
 0 & \text{if } k \text{ is even} \\
 \frac{4}{k\pi} & \text{if } k \text{ is odd} 
\end{cases}
\]

That is

\[
 F^o(x) = \sum_{k=1}^{\infty} \frac{1}{k\pi} \sin(kx)
\]
Because it is an even periodic function with period $2\ell = 2\pi$, $F^e(x)$ has the Fourier series expansion

$$F^e(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \left(\frac{k\pi x}{\ell}\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx)$$

with

$$a_k = \frac{2}{\ell} \int_0^\ell F^e(x) \cos \left(\frac{k\pi x}{\ell}\right) \, dx = \frac{2}{\pi} \int_0^\pi 1 \cos(kx) \, dx = \begin{cases} 2 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

That is,

$$F^e(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) = 1 \quad \text{(surprise!)}$$

For all $0 < x < \pi$ we have both

$$f(x) = F^e(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) = 1$$

$$f(x) = F^o(x) = \sum_{k=1}^{\infty} b_k \sin(kx) = \sum_{k=1}^{\infty} \frac{4}{k\pi} \sin(kx)$$