

Periodic Extensions

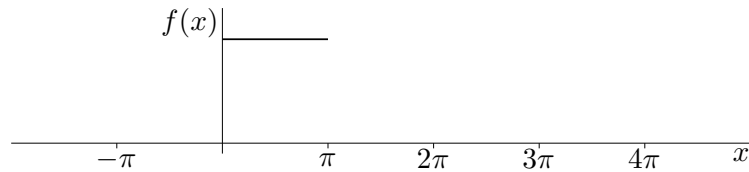
We know that every (sufficiently smooth) periodic function has a Fourier series expansion. It is fairly common for functions arising from certain applications to be defined only on a finite interval $0 < x < \ell$. This is the case if, for example, $f(x)$ is the vertical displacement of a string from the x axis at position x and if the string only runs from $x = 0$ to $x = \ell$. For the application, we only care about x 's between 0 and ℓ . But we are free to extend the definition of $f(x)$ to $x < 0$ and $x > \ell$, for computational purposes.

We define $F(x)$ to be a **periodic extension** of $f(x)$ if

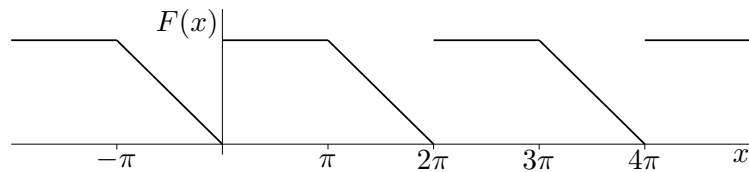
$$i) F(x) = f(x) \text{ for } 0 < x < \ell$$

$$ii) F(x) \text{ is periodic of period } 2\ell$$

There are many different periodic extensions of $f(x)$. Most of them are pretty useless. For example define $f(x) = 1$ for $0 < x < \pi$. We are leaving $f(x)$ undefined for $x \geq \pi$ or $x \leq 0$.



Then the function $F(x)$ graphed in



is a perfectly legitimate, but virtually useless, periodic extension of f .

There are two periodic extensions for $f(x)$ that are very useful. Given any $f(x)$, which is defined only for $0 < x < \ell$, we define its **even periodic extension**, $F^e(x)$, by the conditions that

- a) $F^e(x) = f(x)$ for $0 < x < \ell$
- b) $F^e(x)$ is even (This fixes $F^e(x)$ for $-\ell < x < 0$.)
- c) $F^e(x)$ has period 2ℓ (This fixes $F^e(x)$ for all remaining x 's, except $x = n\pi$.)

and we define its **odd periodic extension**, $F^o(x)$, by

- a) $F^o(x) = f(x)$ for $0 < x < \ell$
- b) $F^o(x)$ is odd
- c) $F^o(x)$ has period 2ℓ

Because F^e is even, it has a Fourier cosine expansion. Because F^o is odd, it has a Fourier sine expansion. We have arranged that $f(x) = F^e(x) = F^o(x)$ for all $0 < x < \ell$. So, provided we

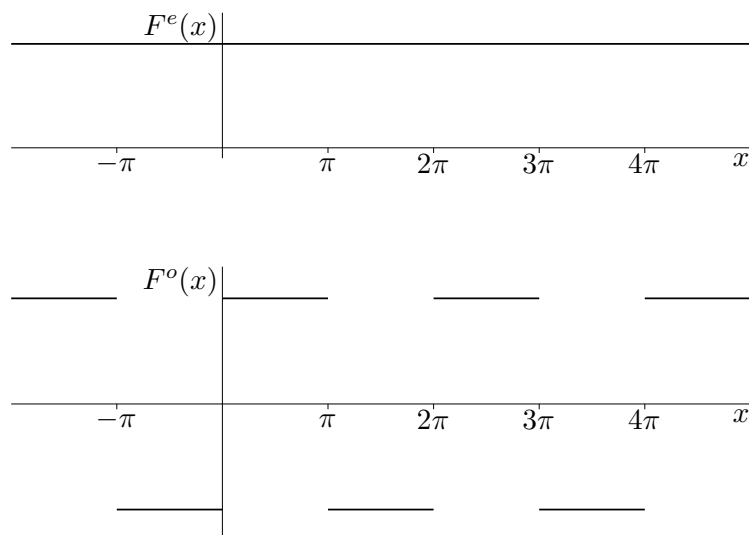
restrict to $0 < x < \ell$, we have

$$\begin{aligned} f(x) &= F^e(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{\ell}\right) \\ &= F^o(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{\ell}\right) \end{aligned}$$

where

$$\begin{aligned} a_k &= \frac{2}{\ell} \int_0^{\ell} F^e(x) \cos\left(\frac{k\pi x}{\ell}\right) dx = \frac{2}{\ell} \int_0^{\ell} f(x) \cos\left(\frac{k\pi x}{\ell}\right) dx \\ b_k &= \frac{2}{\ell} \int_0^{\ell} F^o(x) \sin\left(\frac{k\pi x}{\ell}\right) dx = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{k\pi x}{\ell}\right) dx \end{aligned}$$

For example consider, again, the function $f(x)$ which is only defined for $0 < x < \pi$ and takes the value $f(x) = 1$ for all $0 < x < \pi$. The even and odd periodic extensions, $F^e(x)$ and $F^o(x)$ of this function are graphed below.



Both $F^e(x)$ and $F^o(x)$ take the value 1 for all $0 < x < \pi$. Both $F^e(x)$ and $F^o(x)$ have period 2π . But $F^e(x)$ is an even function while $F^o(x)$ is an odd function. Because it is an odd periodic function with period $2\ell = 2\pi$, $F^o(x)$ has the Fourier series expansion

$$F^o(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{\ell}\right) = \sum_{k=1}^{\infty} b_k \sin(kx)$$

with

$$b_k = \frac{2}{\ell} \int_0^{\ell} F^o(x) \sin\left(\frac{k\pi x}{\ell}\right) dx = \frac{2}{\pi} \int_0^{\pi} 1 \sin(kx) dx = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{4}{k\pi} & \text{if } k \text{ is odd} \end{cases}$$

That is

$$F^o(x) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{k\pi} \sin(kx)$$

Because it is an even periodic function with period $2\ell = 2\pi$, $F^e(x)$ has the Fourier series expansion

$$F^e(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{\ell}\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx)$$

with

$$a_k = \frac{2}{\ell} \int_0^{\ell} F^e(x) \cos\left(\frac{k\pi x}{\ell}\right) dx = \frac{2}{\pi} \int_0^{\pi} 1 \cos(kx) dx = \begin{cases} 2 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

That is,

$$F^e(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) = 1 \quad (\text{surprise!})$$

For all $0 < x < \pi$ we have both

$$f(x) = F^e(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) = 1$$

$$f(x) = F^o(x) = \sum_{k=1}^{\infty} b_k \sin(kx) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4}{k\pi} \sin(kx)$$