Fourier Series

Much of this course concerns the problem of representing a function as a sum of its different frequency components. Expressing a musical tone as a sum of a fundamental tone and various harmonics is such a representation. So is a spectral decomposition of light waves. The ability to isolate the signal of a single radio or television station from the dozens that are being simultaneously received depends on being able to amplify certain frequencies and suppress others.

We start our look at the theory of Fourier series with two questions:

Question #1

Which functions $f(t)$ have a representation as a sum of constants times $\cos(kt)$'s and $\sin(kt)$'s? Since $\cos(kt)$ and $\sin(kt)$ can be written in terms of complex exponentials$^{(1)}$ using

$$\cos(kt) = \tfrac{1}{2}\big[e^{ikt} + e^{-ikt}\big] \qquad \sin(kt) = \tfrac{1}{2i}\big[e^{ikt} - e^{-ikt}\big]$$

and, conversely, $e^{\pm ikt}$ can be written in terms of $\cos(kt)$'s and $\sin(kt)$'s using

$$e^{\pm ikt} = \cos(kt) \pm i\sin(\pm kt)$$

it is equivalent to ask which functions $f(t)$ have a representation

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \tag{1}$$

for some constants c_k. Because it will save us considerable writing we shall start with this form of the question and return to sines and cosines later.

First, observe that, for every integer k, $e^{ikt} = \cos(kt) + i\sin(kt)$ is periodic with period 2π. So the right hand side is necessarily periodic of period 2π. Unless $f(t)$ is periodic with period 2π, it cannot possibly have a representation of the form (1). We shall shortly state a result that says that, on the other hand, every sufficiently continuous (details later) function of period 2π has a representation (1). In this course, we shall never fully justify this claim. On the other hand, it is fairly easy justify an analogous claim for "Discrete Fourier Series", which is the version of Fourier series for functions $f(t)$ that are only defined for $t = n\tau$, with n running over the integers and τ a fixed spacing. This is done in the notes "Discrete–Time Fourier Series and Fourier Transforms". Before giving the detailed answer to this question, we consider

Question #2

Suppose that we know that some specific function $f(t)$ has a representation of the form (1). What are the values of the coefficients c_k? With a little trickery, we shall be able to answer this question completely and easily. We wish to solve (1) for the c_k's. At first this task looks somewhat daunting because (1) is really a system of infinitely many equations (one equation for each value of t) in infinitely many unknowns (the c_k's). The trick will allow us to reduce this system to a single equation in any one unknown. Suppose, for example, that we wish to solve for c_{17}. The index 17 has been chosen at random. Then we use the "orthogonality relation" that, when $k \neq 17$,

$$\int_{-\pi}^{\pi} e^{ikt}e^{-i17t}\,dt = \int_{-\pi}^{\pi} e^{i(k-17)t}\,dt = \tfrac{1}{i(k-17)}e^{i(k-17)t}\Big|_{-\pi}^{\pi} = \tfrac{1}{i(k-17)}\big[e^{i(k-17)\pi} - e^{-i(k-17)\pi}\big] = 0 \tag{2}$$

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(1) See the notes entitled "Complex Numbers and Exponentials"

(because $e^{i(k-17)\pi}/e^{-i(k-17)\pi} = e^{i(k-17)2\pi} = 1$ so that $e^{i(k-17)\pi} = e^{-i(k-17)\pi}$ for all integers k) to eliminate all the c_k's with $k \neq 17$ from (1). To do so, take (1), multiply it by e^{-i17t} and integrate the result from $-\pi$ to π. This gives

$$\int_{-\pi}^{\pi} f(t)e^{-i17t}\ dt = \sum_{k=-\infty}^{\infty} c_k \int_{-\pi}^{\pi} e^{ikt}e^{-i17t}\ dt$$

Because of (2), all of the terms on the right hand side with $k \neq 17$ are zero. Thus

$$\int_{-\pi}^{\pi} f(t)e^{-i17t}\ dt = c_{17}\int_{-\pi}^{\pi} e^{i17t}e^{-i17t}\ dt = c_{17}\int_{-\pi}^{\pi}\ dt = 2\pi\ c_{17}$$

As promised, this is a single equation in the single unknown c_{17}, which we can trivially solve.

$$c_{17} = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-i17t}\ dt$$

Of course, we could have done the same thing with the integer 17 replaced by any other integer m. This would have given

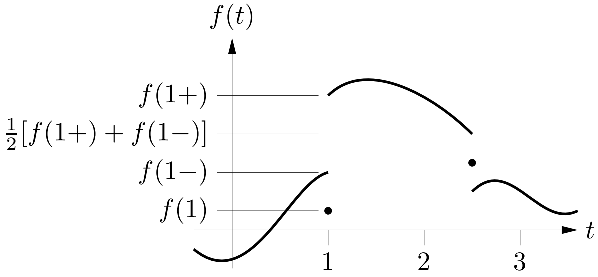
$$c_m = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-imt}\ dt \tag{3}$$

We now know all of the Fourier coefficients and are ready to return to the answer to question #1.

The Main Fourier Series Expansion.

We shall shortly construct a number of different Fourier series expansions that are used for various different classes of functions. For all of these expansions, we are going to restrict our attention to functions that are piecewise continuous with piecewise continuous first derivative. In applications, most functions satisfy these regularity requirements. We start with the definition of "piecewise continuous".

A function $f(t)$ is said to be piecewise continuous if it is continuous except for isolated jump discontinuities. In the example below, $f(t)$ is continuous except for jump discontinuities at $t = 1$ and $t = 2.5$. If a function $f(t)$ has a jump discontinuity at t_0, then the value of $f(t)$ as it approaches t_0 from the left and from the right are still well–defined. These values are



$$f(t_0-) = \lim_{\substack{t\to t_0 \\ t<t_0}} f(t) \qquad \text{and} \qquad f(t_0+) = \lim_{\substack{t\to t_0 \\ t>t_0}} f(t)$$

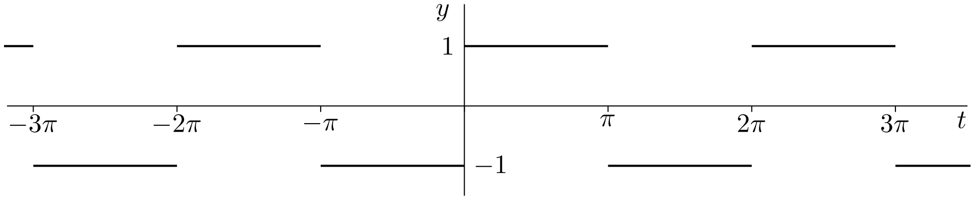
respectively. If f were continuous at t_0, we would have $f(t_0) = f(t_0+) = f(t_0-)$. At a jump, however, there is no a priori relation between $f(t_0)$, $f(t_0-)$ and $f(t_0+)$. In the example above, $f(1)$ is well below both $f(1-)$ and $f(1+)$. On the other hand, it is fairly common for the value of f at the jump t_0 to be precisely at the midpoint of the jump. That is $f(t_0) = \frac{1}{2}[f(t_0+) + f(t_0-)]$. In the example, this is the case at $t_0 = 2.5$.

Theorem 1 (Fourier Series) *Let $f(t)$ be piecewise continuous with piecewise continuous first derivative and also be periodic with period 2π. Then*

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt} = \begin{cases} f(t) & \text{if } f \text{ is continuous at } t \\ \frac{f(t+)+f(t-)}{2} & \text{otherwise} \end{cases}$$

for all $-\infty < t < \infty$ if and only if $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt$ for all integers k.

Example 2 As a first example, we consider the function $f(t)$ whose graph appears in the figure below.



According to our main Fourier series theorem

$$c_k = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt = \tfrac{1}{2\pi}\int_{0}^{\pi} e^{-ikt}\,dt + \tfrac{1}{2\pi}\int_{-\pi}^{0} (-1)e^{-ikt}\,dt$$

For $k = 0$

$$c_k = \tfrac{1}{2\pi}\int_{0}^{\pi} dt - \tfrac{1}{2\pi}\int_{-\pi}^{0} dt = \tfrac{1}{2\pi}\pi - \tfrac{1}{2\pi}\pi = 0$$

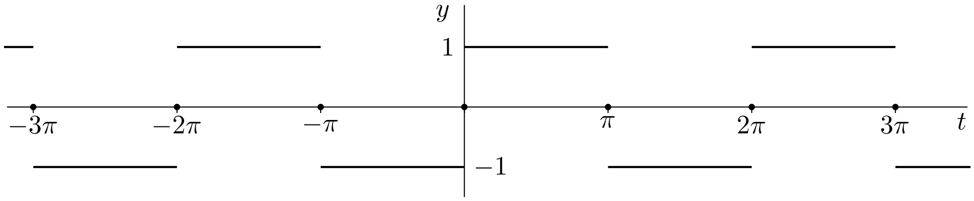
For $k \neq 0$

$$c_k = \tfrac{1}{2\pi}\left[\tfrac{1}{-ik}e^{-ikt}\right]_0^{\pi} - \tfrac{1}{2\pi}\left[\tfrac{1}{-ik}e^{-ikt}\right]_{-\pi}^{0} = \tfrac{i}{2k\pi}\left[e^{-ik\pi} - 1 - 1 + e^{-ik(-\pi)}\right]$$

Since $e^{ik\pi}$ and $e^{-ik\pi}$ are both -1 for k odd and $+1$ for k even

$$c_k = \begin{cases} -\frac{2}{k\pi}i & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

The Fourier series theorem tells us that the graph of $\sum_{k \text{ odd}} \frac{2}{k\pi i}e^{ikt}$ is



Example 3 The function

$$f(t) = \sin\left(t + \tfrac{\pi}{4}\right)$$

is periodic of period 2π and so has a Fourier expansion. We could compute the Fourier coefficients c_k by evaluating the integral

$$c_k = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} \sin\left(t + \tfrac{\pi}{4}\right)e^{-ikt}\,dt \tag{4}$$

But it is much easier to simply observe that

$$\begin{aligned} f(t) &= \tfrac{1}{2i}\left(e^{i(t+\frac{\pi}{4})} - e^{-i(t+\frac{\pi}{4})}\right) = \tfrac{1}{2i}e^{i\frac{\pi}{4}}e^{it} - \tfrac{1}{2i}e^{-i\frac{\pi}{4}}e^{-it} = \tfrac{1}{2i}\tfrac{1}{\sqrt{2}}(1+i)e^{it} - \tfrac{1}{2i}\tfrac{1}{\sqrt{2}}(1-i)e^{-it} \\ &= \tfrac{1}{2\sqrt{2}}(1-i)e^{it} + \tfrac{1}{2\sqrt{2}}(1+i)e^{-it} \end{aligned} \tag{5}$$

This is of the form $\sum_{k=-\infty}^{\infty} c_k e^{ikt}$ with

$$c_k = \begin{cases} \frac{1}{2\sqrt{2}}(1-i) & \text{if } k=1 \\ \frac{1}{2\sqrt{2}}(1+i) & \text{if } k=-1 \\ 0 & \text{if } k \ne -1, 1 \end{cases}$$

The Fourier coefficients of a periodic function are unique. We showed this in the course of answering Question # 2. So the right hand side of (5) is *THE* Fourier expansion for $f(t) = \sin\big(t + \frac{\pi}{4}\big)$. There is no need to evaluate the integrals in (4). Not only would that be a lot of wasted effort, but we would probably make a mechanical error along the way and end up with the wrong answer.

Example 4 Suppose that the function

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \tag{6}$$

is real–valued for all $-\infty < t < \infty$. What conclusions can be drawn concerning the Fourier coefficients c_k? A number is real if and only if it is the same as its complex conjugate. So $f(t)$ is real if and only if $f(t) = \overline{f(t)}$. Substituting in (6) gives that $f(t)$ is real if and only if

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt} = \sum_{k=-\infty}^{\infty} \overline{c_k} e^{-ikt} = \sum_{k=-\infty}^{\infty} \overline{c_{-k}} e^{ikt}$$

For the second equality, we replaced k by $-k$ in the sum. This is legitimate because, as k runs over all of the integers from $-\infty$ to $+\infty$, $-k$ also runs exactly once over all integers. Again by the uniqueness of Fourier coefficients (the "only if" part of Theorem 1), the two sums are equal for all t if and only if $c_k = \overline{c_{-k}}$ for all k. So we conclude that

$$f(t) \text{ is real for all } -\infty < t < \infty \iff c_k = \overline{c_{-k}} \text{ for all } -\infty < k < \infty$$

For example

$$\sin t = \tfrac{1}{2i}\big[e^{it} - e^{-it}\big] = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \text{ with } c_k = \begin{cases} \frac{1}{2i} & \text{if } k=1 \\ -\frac{1}{2i} & \text{if } k=-1 \\ 0 & \text{otherwise} \end{cases}$$

is a real–valued function and c_1 and c_{-1} are indeed complex conjugates of each other.

Example 5 The function

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \tag{7}$$

is necessarily periodic of period 2π. Suppose that it is also periodic of period π. For example, the function $\cos(2t)$ is periodic of period 2π and also periodic of period π. What conclusions can be drawn concerning the Fourier coefficients c_k? The function $f(t)$ has period π if and only if $f(t) = f(t+\pi)$ for all t. Substituting in (7) gives that $f(t)$ has period π if and only if

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt} = \sum_{k=-\infty}^{\infty} c_k e^{ik(t+\pi)} = \sum_{k=-\infty}^{\infty} \big(c_k e^{ik\pi}\big) e^{ikt}$$

Once again by the uniqueness of Fourier coefficients, the two sums are equal for all t if and only if $c_k = c_k e^{ik\pi}$ for all k. For k even, $e^{ik\pi} = 1$ and $c_k = c_k e^{ik\pi}$ is trivially true. But for k odd, $e^{ik\pi} = -1$ and $c_k = c_k e^{ik\pi}$ is true if and only if $c_k = 0$. So we conclude that

$$f(t) \text{ has period } \pi \iff c_k = 0 \text{ for all odd } k$$

For example

$$\cos(2t) = \tfrac{1}{2}\big[e^{i2t} + e^{-2it}\big] = \sum_{k=-\infty}^{\infty} c_k e^{ikt} \text{ with } c_k = \begin{cases} \frac{1}{2} & \text{if } k = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

does indeed have $c_k = 0$ for all odd k.

Using Theorem 1, we can easily come up with lots of variations. In these variations, we shall assume that all functions are piecewise continuous with piecewise continuous first derivative and we shall also assume that $f(t) = \frac{f(t+)+f(t-)}{2}$ for all t.

Variation #1 – period 2ℓ

It is easy to modify our Fourier series result to apply to functions that have period 2ℓ, for some $\ell > 0$, rather than 2π. Just rename the variable t in the Fourier Series Theorem to τ and then make the change of variables $\tau = \frac{\pi}{\ell}t$. This gives

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{\ell}t} \qquad \text{with} \qquad c_k = \tfrac{1}{2\ell}\int_{-\ell}^{\ell} f(t)e^{-ik\frac{\pi}{\ell}t}\, dt \tag{8}$$

As a check, note that since $e^{ik\frac{\pi}{\ell}(t+2\ell)} = e^{ik\frac{\pi}{\ell}t}e^{i2k\pi} = e^{ik\frac{\pi}{\ell}t}$, so that $e^{ik\frac{\pi}{\ell}t}$ has period 2ℓ. The formula for c_k in (8) can be derived directly, using

$$\int_{-\ell}^{\ell} e^{ik\frac{\pi}{\ell}t}e^{-im\frac{\pi}{\ell}t}\, dt = \begin{cases} 2\ell & \text{if } k = m \\ 0 & \text{if } k \ne m \end{cases}$$

and the same strategy as led to (3).

Variation #2 – $\sin$'s and $\cos$'s

To convert (8) into sin's and cos's just sub in $e^{ik\frac{\pi}{\ell}t} = \cos(\frac{k\pi t}{\ell}) + i\sin(\frac{k\pi t}{\ell})$.

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{\ell}t} = \sum_{k=-\infty}^{\infty} c_k\big[\cos(\tfrac{k\pi t}{\ell}) + i\sin(\tfrac{k\pi t}{\ell})\big]$$

The $k = 0$ term in this sum is

$$c_0\big[\cos(\tfrac{0\pi t}{\ell}) + i\sin(\tfrac{0\pi t}{\ell})\big] = c_0$$

The $k = 1$ and $k = -1$ terms together are

$$c_1\big[\cos(\tfrac{\pi t}{\ell}) + i\sin(\tfrac{\pi t}{\ell})\big] + c_{-1}\big[\cos(-\tfrac{\pi t}{\ell}) + i\sin(-\tfrac{\pi t}{\ell})\big] = [c_1 + c_{-1}]\cos(\tfrac{\pi t}{\ell}) + [ic_1 - ic_{-1}]\sin(\tfrac{\pi t}{\ell})$$

since $\cos(-\frac{\pi t}{\ell}) = \cos(\frac{\pi t}{\ell})$ and $\sin(-\frac{\pi t}{\ell}) = -\sin(\frac{\pi t}{\ell})$. Similarly, the $k = 2$ and $k = -2$ terms together are

$$c_2\big[\cos(\tfrac{2\pi t}{\ell}) + i\sin(\tfrac{2\pi t}{\ell})\big] + c_{-2}\big[\cos(-\tfrac{2\pi t}{\ell}) + i\sin(-\tfrac{2\pi t}{\ell})\big] = [c_2 + c_{-2}]\cos(\tfrac{2\pi t}{\ell}) + [ic_2 - ic_{-2}]\sin(\tfrac{2\pi t}{\ell})$$

and so on. Hence

$$f(t) = c_0 + [c_1 + c_{-1}]\cos(\tfrac{\pi t}{\ell}) + [ic_1 - ic_{-1}]\sin(\tfrac{\pi t}{\ell}) + [c_2 + c_{-2}]\cos(\tfrac{2\pi t}{\ell}) + [ic_2 - ic_{-2}]\sin(\tfrac{2\pi t}{\ell}) + \cdots$$

It is conventional to rename c_0 to $\frac{a_0}{2}$. The extra $\frac{1}{2}$ will make some later formulae cleaner. It is also conventional to rename $[c_1 + c_{-1}]$ to a_1, $[c_2 + c_{-2}]$ to a_2, etc. and $[ic_1 - ic_{-1}]$ to b_1, $[ic_2 - ic_{-2}]$ to b_2, etc. Note that nobody said that c_2 or c_{-2} had to be a real number. So it is perfectly possible for b_2 to be a real number. In fact, it usually is. With these new names

$$f(t) = \tfrac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k\cos\left(\tfrac{k\pi t}{\ell}\right) + b_k\sin\left(\tfrac{k\pi t}{\ell}\right)\right] \tag{9}$$

The coefficients (including a_0, because $a_0 = 2c_0$) are determined by

$$\begin{aligned} a_k = c_k + c_{-k} \quad &= \tfrac{1}{2\ell}\int_{-\ell}^{\ell} f(t)\ \left[e^{-ik\frac{\pi}{\ell}t} + e^{ik\frac{\pi}{\ell}t}\right]\ dt \ = \tfrac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos\left(\tfrac{k\pi t}{\ell}\right)\ dt \\ b_k = i[c_k - c_{-k}] &= \tfrac{1}{2\ell}\int_{-\ell}^{\ell} f(t)\ i\left[e^{-ik\frac{\pi}{\ell}t} - e^{ik\frac{\pi}{\ell}t}\right]\ dt = \tfrac{1}{\ell}\int_{-\ell}^{\ell} f(t)\sin\left(\tfrac{k\pi t}{\ell}\right)\ dt \end{aligned} \tag{10}$$

Variation #3 – f odd

If $f(t)$ is an odd function, that is if $f(-t) = -f(t)$ like $\sin t$, then $f(t)\cos\left(\frac{k\pi t}{\ell}\right)$ is an odd function and

$$a_k = \tfrac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos(\tfrac{k\pi t}{\ell})\ dt = 0$$

for all k. Also $f(t)\sin\left(\frac{k\pi t}{\ell}\right)$ is an even function so that

$$b_k = \tfrac{1}{\ell}\int_{-\ell}^{\ell} f(t)\sin(\tfrac{k\pi t}{\ell})\ dt = \tfrac{2}{\ell}\int_0^{\ell} f(t)\sin(\tfrac{k\pi t}{\ell})\ dt$$

So if f has period 2ℓ and is also odd, our Fourier series expansion simplifies to

$$f(t) = \sum_{k=1}^{\infty} b_k\sin\left(\tfrac{k\pi t}{\ell}\right) \qquad \text{with} \qquad b_k = \tfrac{2}{\ell}\int_0^{\ell} f(t)\sin\left(\tfrac{k\pi t}{\ell}\right)\ dt \tag{11}$$

Variation #4 – f even

If $f(t)$ is an even function, that is if $f(-t) = f(t)$ like $\cos t$, then $f(t)\sin\left(\frac{k\pi t}{\ell}\right)$ is an odd function and

$$b_k = \tfrac{1}{\ell}\int_{-\ell}^{\ell} f(t)\sin(\tfrac{k\pi t}{\ell})\ dt = 0$$

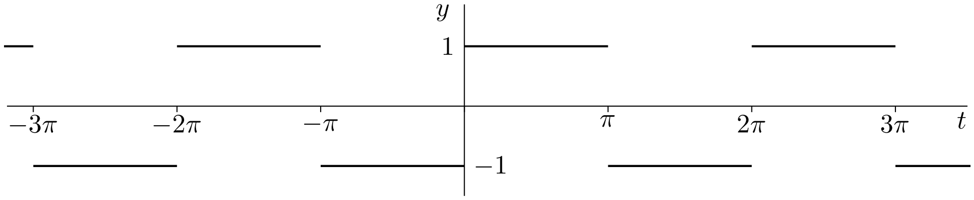
for all k. Also $f(t)\cos\left(\frac{k\pi t}{\ell}\right)$ is an even function so that

$$a_k = \tfrac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos(\tfrac{k\pi t}{\ell})\ dt = \tfrac{2}{\ell}\int_0^{\ell} f(t)\cos(\tfrac{k\pi t}{\ell})\ dt$$

So if f has period 2ℓ and is also even, our Fourier series expansion simplifies to

$$f(t) = \tfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos\left(\tfrac{k\pi t}{\ell}\right) \qquad \text{with} \qquad a_k = \tfrac{2}{\ell}\int_0^{\ell} f(t)\cos(\tfrac{k\pi t}{\ell})\ dt \tag{12}$$

Example 6 Consider once again the function of Example 2. The graph of that function is repeated in the figure below. This function has period 2π, takes the value -1 for $-\pi < t < 0$ and the value $+1$ for $0 < t < 1$.



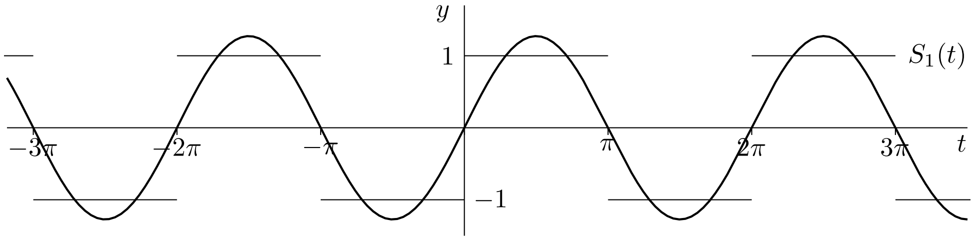
This function is also odd, so it has a Fourier sin series expansion (11) with $\ell = \pi$. The Fourier coefficient

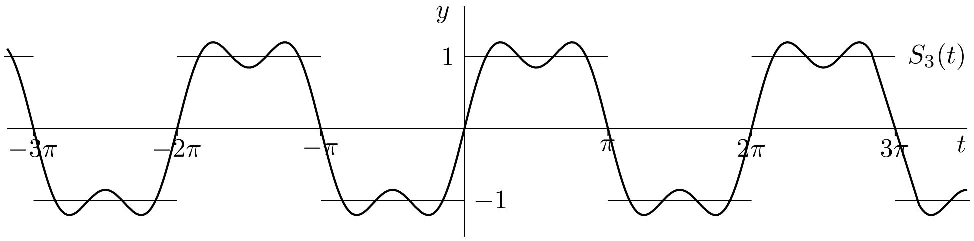
$$b_k = \tfrac{2}{\ell}\int_0^\ell f(t)\sin\!\left(\tfrac{k\pi t}{\ell}\right)\,dt = \tfrac{2}{\pi}\int_0^\pi f(t)\sin(kt)\,dt = \tfrac{2}{\pi}\int_0^\pi \sin(kt)\,dt = -\tfrac{2}{k\pi}\cos(kt)\Big|_0^\pi = -\tfrac{2}{k\pi}\left[(-1)^k - 1\right]$$
$$= \begin{cases} \frac{4}{k\pi} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

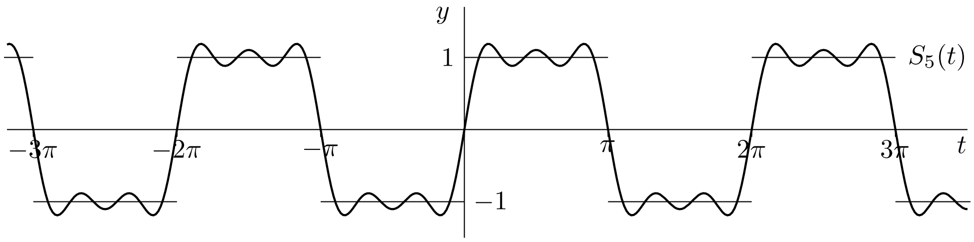
The Fourier series expansion for the function $f(t)$ graphed above is

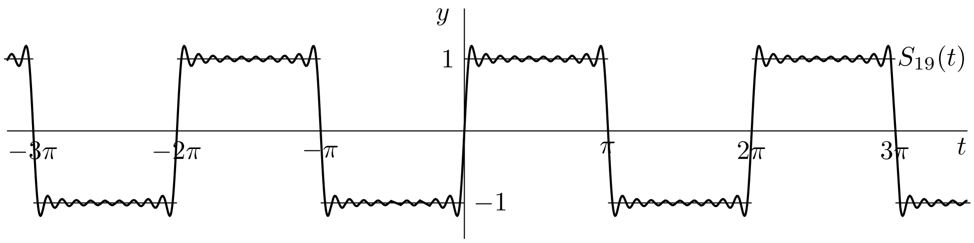
$$f(t) = \sum_{k=1}^\infty b_k \sin(kt) = \sum_{\substack{k=1 \\ k \text{ odd}}}^\infty \tfrac{4}{k\pi}\sin(kt)$$

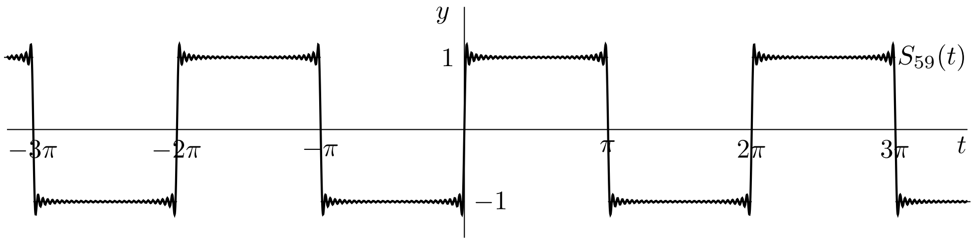
One first observation to make is that whenever t is at a jump discontinuity of $f(t)$, i.e. whenever t is an integer multiple of π, then $\sin(kt) = 0$ for all k and so $\sum_{\substack{k=1 \\ k \text{ odd}}}^\infty \frac{4}{k\pi}\sin(kt) = 0$, right in the middle of the jump, as it is supposed to be. To give some idea of how good the Fourier series expansion works, I have graphed below a number of partial sums $S_N(t) = \sum_{\substack{1\le k\le N \\ k \text{ odd}}} \frac{4}{k\pi}\sin(kt)$. The first graph is of $S_1(t) = \frac{4}{\pi}\sin(t)$ and is not a very good likeness of $f(t)$. The second, $S_3(t) = \frac{4}{\pi}\sin(t) + \frac{4}{3\pi}\sin(3t)$, is already starting to look a little like $f(t)$. As we add more and more terms the graphs start looking more and more like $f(t)$, except that









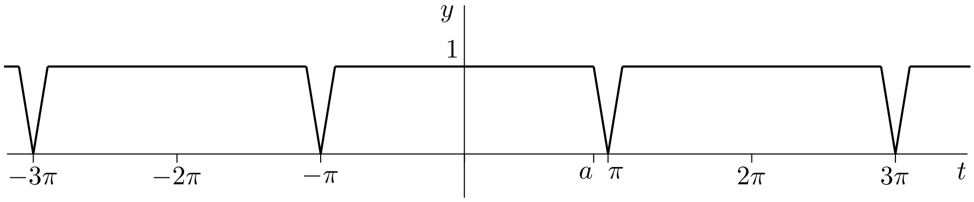


they exhibit a little "ringing" right at the discontinuities. This "ringing" is always present in partial sums of Fourier series at jump discontinuities. It is called the Gibbs phenomenon.

Example 7 As another example, we replace the jump discontinuity of the first example by a ramp. We shall see that, even if the ramp is moderately steep, the ringing of Gibbs phenomenon disappears. Just to put in an additional change, I'll make the function even rather than odd. Its graph is given in the figure below. This function has period 2π, takes the value 1 for $0 < t < a$ and decreases from 1 down to 0 and t runs from a up to π. For $0 \le t \le \pi$, the function is given by the formula

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t \le a \\ \frac{\pi - t}{\pi - a} & \text{if } a \le t \le \pi \end{cases}$$

This function is also even, so it has a Fourier cosine series expansion (12) with $\ell = \pi$. The Fourier coefficient,



for $k \ne 0$, is

$$a_k = \tfrac{2}{\ell}\int_0^\ell f(t)\cos\!\left(\tfrac{k\pi t}{\ell}\right)\,dt = \tfrac{2}{\pi}\int_0^\pi f(t)\cos(kt)\,dt = \tfrac{2}{\pi}\int_0^a \cos(kt)\,dt + \tfrac{2}{\pi}\int_a^\pi \tfrac{\pi - t}{\pi - a}\cos(kt)\,dt$$

To evaluate these integrals we need the indefinite integrals of $\cos(kt)$ and $t\cos(kt)$. The first one is trivial

$$\int \cos(kt)\,dt = \tfrac{1}{k}\sin(kt) + C$$

The second is normally computed using integration by parts. But it is easier to just apply $\frac{d}{dk}$ to both sides of

$$\int \sin(kt)\,dt = -\tfrac{1}{k}\cos(kt) + C$$

Note that, while we only need this integral for integer k, it is valid for all nonzero k. So it is legitimate to differentiate with respect to k. Also note that, while the "constant" C is independent of t, it is allowed to depend on k, so its derivative with respect to k need not be zero. In any event, applying $\frac{d}{dk}$ gives

$$\int t\cos(kt)\ dt = \tfrac{t}{k}\sin(kt) + \tfrac{1}{k^2}\cos(kt) + C'$$

so that

$$\begin{aligned} a_k &= \tfrac{2}{k\pi}\sin(kt)\Big|_0^a + \tfrac{2}{k(\pi-a)}\sin(kt)\Big|_a^\pi - \tfrac{2}{\pi(\pi-a)}\Big[\tfrac{t}{k}\sin(kt) + \tfrac{1}{k^2}\cos(kt)\Big]_a^\pi \\ &= \Big[\tfrac{2}{k\pi} - \tfrac{2}{k(\pi-a)}\Big]\sin(ka) - \tfrac{2(-1)^k}{k^2\pi(\pi-a)} + \tfrac{2}{\pi(\pi-a)}\Big[\tfrac{a}{k}\sin(ka) + \tfrac{1}{k^2}\cos(ka)\Big] \end{aligned}$$

For $k = 0$,

$$a_0 = \tfrac{2}{\ell}\int_0^\ell f(t)\cos\big(\tfrac{0\pi t}{\ell}\big)\ dt = \tfrac{2}{\pi}\int_0^\pi f(t)\ dt$$

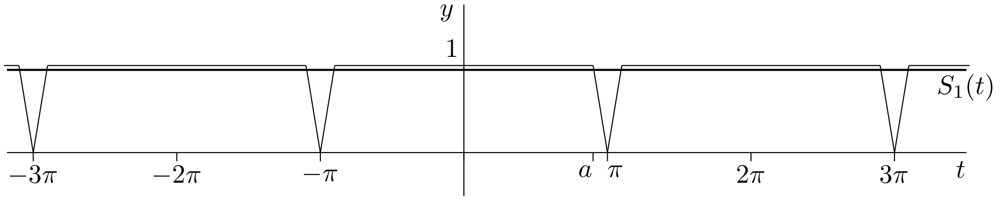
The integral is the area under the graph for $0 \le t \le \pi$. The region under the graph consists of a rectangle of height 1 and base a and a triangle of height 1 and base $\pi - a$. So

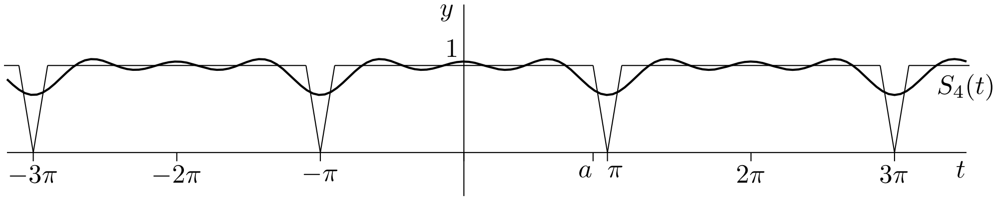
$$a_0 = \tfrac{2}{\pi}\big[a + \tfrac{1}{2}(\pi - a)\big] = 1 + \tfrac{a}{\pi}$$

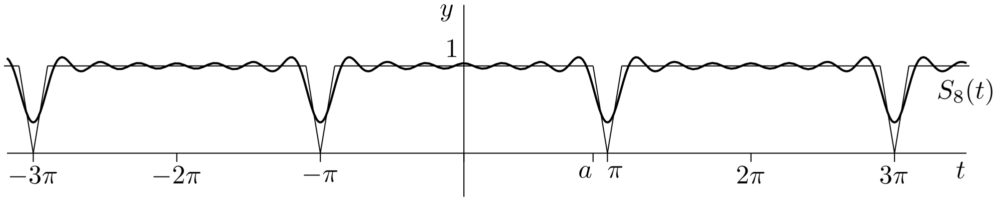
The Fourier series expansion for the function $f(t)$ graphed above is

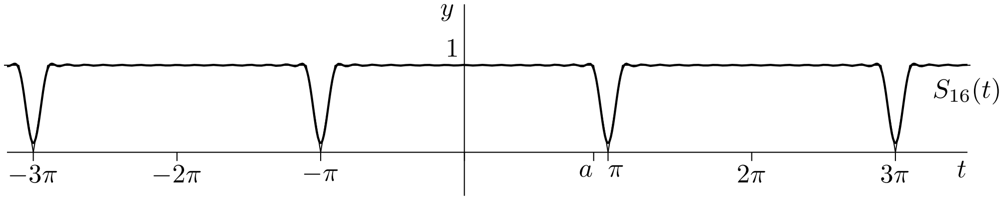
$$f(t) = \tfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kt)$$

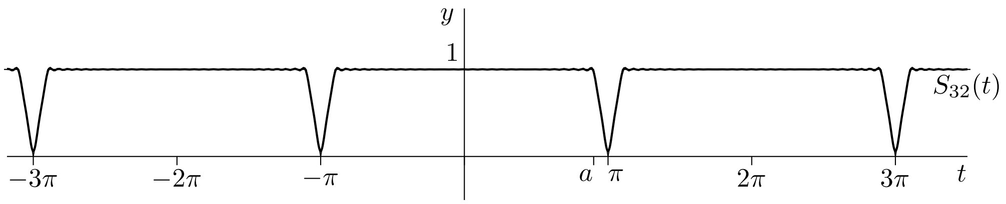
with the a_k's given above. these a_k's are a little messy to compute by hand, but easy to program. Once again, I have had my computer graph a number of partial sums $S_N(t) = \frac{a_0}{2} + \sum_{k=1}^{N-1} a_k\cos(kt)$, for a specific choice of the ramp parameter a, namely $a = 0.9\pi$. The first graph is of $S_1(t) = 1 + \frac{a}{\pi}$ is just a straight line whose height is the average value of $f(t)$. This time we get a really good likeness of the graph of $f(t)$ before we have included a hundred terms.

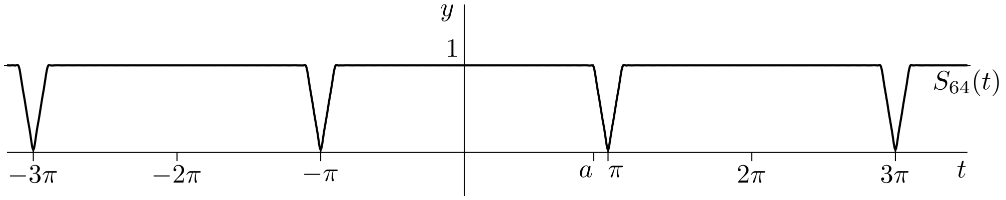












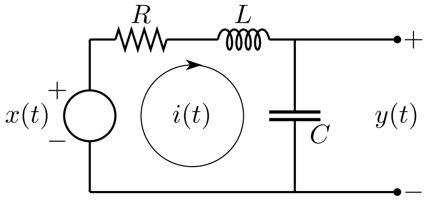
Example 8 By a standard trig identity, the periodic function $f(t) = \cos^2 t$ obeys

$$f(t) = \cos^2 t = \tfrac{1}{2} + \tfrac{1}{2}\cos(2t) \tag{13}$$

The right hand side is a Fourier cosine expansion $\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt)$ with $a_0 = 1$, $a_2 = \frac{1}{2}$ and all other a_k's zero. Just as in the complex case, the coefficients in the real Fourier expansions are uniquely determined. So the right hand side of (13) is the Fourier expansion for $f(t) = \cos^2 t$.

Application of Fourier Series to Ordinary Differential Equations.

Consider the RLC circuit



We're going to think of the voltage $x(t)$ as a input signal and the voltage $y(t)$ as an output signal. The goal is to determine the output voltage for any given input voltage. If $i(t)$ is the current flowing at time t in the loop as shown and $q(t)$ is the charge on the capacitor, then the voltage across R, L and C, respectively, at time t are $Ri(t)$, $L\frac{di}{dt}(t)$ and $y(t) = \frac{q(t)}{C}$. By Kirchhoff's law, these three voltages must add up to $x(t)$ so that

$$Ri(t) + L\tfrac{di}{dt}(t) + \tfrac{q(t)}{C} = x(t) \tag{14}$$

This is one equation in the two unknowns, $i(t)$ and $q(t)$. Fortunately, there is a relationship between the two. Namely

$$i(t) = \tfrac{dq}{dt}(t) = Cy'(t) \tag{15}$$

This just says that the capacitor cannot create or destroy charge on its own. All charging of the capacitor must come from the current. Subbing (15) into (14) gives

$$LCy''(t) + RCy'(t) + y(t) = x(t) \tag{16}$$

We are going to consider the case in which $x(t)$ and $y(t)$ are both$^{(1)}$ periodic$^{(2)}$, say with period 2ℓ. Hence both $x(t)$ and $y(t)$ have Fourier series expansions of the form (9). Electrical engineers routinely use complex Fourier series (8) in place of (9) and we shall do so too. The reason is that the computations tend to be much simpler. Of course, in the real world $x(t)$ and $y(t)$ are real. When electrical engineers write

$$x(t) = \sum_{k=-\infty}^{\infty} \hat{x}_k e^{ik\frac{\pi}{\ell}t} \qquad y(t) = \sum_{k=-\infty}^{\infty} \hat{y}_k e^{ik\frac{\pi}{\ell}t} \tag{17}$$

they really mean that $x(t)$ is the real part of $\sum_{k=-\infty}^{\infty} \hat{x}_k e^{ik\frac{\pi}{\ell}t}$ and $y(t)$ is the real part of $\sum_{k=-\infty}^{\infty} \hat{y}_k e^{ik\frac{\pi}{\ell}t}$. Subbing (17) into (16) gives

$$\sum_{k=-\infty}^{\infty} -LC\tfrac{k^2\pi^2}{\ell^2}\, \hat{y}_k\, e^{ik\frac{\pi}{\ell}t} + \sum_{k=-\infty}^{\infty} iRC\tfrac{k\pi}{\ell}\, \hat{y}_k\, e^{ik\frac{\pi}{\ell}t} + \sum_{k=-\infty}^{\infty} \hat{y}_k\, e^{ik\frac{\pi}{\ell}t} = \sum_{k=-\infty}^{\infty} \hat{x}_k\, e^{ik\frac{\pi}{\ell}t}$$

which is equivalent to

$$\sum_{k=-\infty}^{\infty} \big(-LC\tfrac{k^2\pi^2}{\ell^2} + iRC\tfrac{k\pi}{\ell} + 1\big)\, \hat{y}_k\, e^{ik\frac{\pi}{\ell}t} = \sum_{k=-\infty}^{\infty} \hat{x}_k\, e^{ik\frac{\pi}{\ell}t}$$

The left hand side is a Fourier series expansion with coefficient $c_k = \big(-LC\tfrac{k^2\pi^2}{\ell^2} + iRC\tfrac{k\pi}{\ell} + 1\big)\, \hat{y}_k$. The right hand side is a Fourier series expansion for the same function with coefficient $c_k = \hat{x}_k$. Because the Fourier coefficients of any function are uniquely determined (we saw that in the answer to "Question #2"), the $k^{\rm th}$ Fourier coefficients of the two series must be the same so that

$$\big(-LC\tfrac{k^2\pi^2}{\ell^2} + iRC\tfrac{k\pi}{\ell} + 1\big)\, \hat{y}_k = \hat{x}_k$$

To save writing, let's call the $k^{\rm th}$ frequency $k\frac{\pi}{\ell} = \omega_k$ and let's also write $a_k = 1 - LC\omega_k^2$ and $b_k = RC\omega_k$ so that $\big(-LC\tfrac{k^2\pi^2}{\ell^2} + iRC\tfrac{k\pi}{\ell} + 1\big) = a_k + ib_k$. With this notation

$$\hat{y}_k = \tfrac{1}{a_k+ib_k}\hat{x}_k$$

We conclude that the circuit acts independently on each frequency, ω_k, component of the signal. An input signal $e^{i\omega_k t}$ results in an output signal $\frac{1}{a_k+ib_k}e^{i\omega_k t}$. So a real input signal $\mathrm{Re}\,\big(e^{i\omega_k t}\big) = \cos(\omega_k t)$ results in a real output signal

$$\begin{aligned}\mathrm{Re}\Big\{\tfrac{1}{a_k+ib_k}e^{i\omega_k t}\Big\} &= \mathrm{Re}\Big\{\tfrac{1}{a_k+ib_k}\tfrac{a_k-ib_k}{a_k-ib_k}e^{i\omega_k t}\Big\} = \mathrm{Re}\Big\{\tfrac{a_k-ib_k}{a_k^2+b_k^2}\big[\cos(\omega_k t) + i\sin(\omega_k t)\big]\Big\}\\ &= \tfrac{a_k}{a_k^2+b_k^2}\cos(\omega_k t) + \tfrac{b_k}{a_k^2+b_k^2}\sin(\omega_k t)\end{aligned}$$

because the two other terms, $ia_k\sin(\omega_k t) - ib_k\cos(\omega_k t)$, in the product $(a_k - ib_k)\big[\cos(\omega_k t) + i\sin(\omega_k t)\big]$ are pure imaginary. There is another, easier, way to compute the real part of $\frac{1}{a_k+ib_k}e^{i\omega_k t}$ that also gives a more illuminating form for the answer. First use polar coordinates to write $a_k + ib_k = r_k e^{i\theta_k}$, with $r_k = \sqrt{a_k^2 + b_k^2} = \sqrt{(1 - LC\omega_k^2)^2 + R^2C^2\omega_k^2}$ and $\theta_k = \tan^{-1}\frac{b_k}{a_k} = \tan^{-1}\frac{RC\omega_k}{1-LC\omega_k^2}$. Then

$$\mathrm{Re}\Big\{\tfrac{1}{a_k+ib_k}e^{i\omega_k t}\Big\} = \mathrm{Re}\Big\{\tfrac{1}{r_k e^{i\theta_k}}e^{i\omega_k t}\Big\} = \mathrm{Re}\Big\{\tfrac{1}{r_k}e^{-i\theta_k}e^{i\omega_k t}\Big\} = \mathrm{Re}\Big\{\tfrac{1}{r_k}e^{i(\omega_k t-\theta_k)}\Big\} = \tfrac{1}{r_k}\cos(\omega_k t - \theta_k)$$

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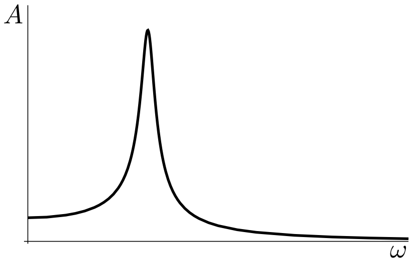
(1) In practice signals are applied to circuits only for finite time intervals. When the input is first turned on, the output usually contains some non-periodic part too. But this non-periodic part is "transient" – meaning that it damps out. We are implicitly assuming that all transients have essentially disappeared from the output signal.

(2) Periodicity is really a bit of a red herring here. A similar procedure, using the Fourier transform (that we will learn about a little later) in place of Fourier series handles non-periodic inputs.

So the RLC cricuit has two effects on the frequency ω_k part of the input signal. Its amplitude is multiplied by $\frac{1}{r_k} = \frac{1}{\sqrt{(1-LC\omega_k^2)^2+R^2C^2\omega_k^2}}$ and it also undergoes a phase shift $\theta_k = \tan^{-1}\frac{RC\omega_k}{1-LC\omega_k^2}$. Here is a graph of $A = \frac{1}{\sqrt{(1-LC\omega^2)^2+R^2C^2\omega^2}}$ against ω



Note that there is a small range of frequencies that give a large amplitude response. This is the phenomenon of resonance. That is why this circuit is often used as a filter to extract a specific frequency from the input signal.

Power and Parseval's Relation

The energy carried by a signal $f(t)$ is $\int_{-\infty}^{\infty}|f(t)|^2\ dt$. For a (nonzero) periodic signal this is always infinite. So for a periodic signal the average power (energy per unit time) is a much more useful quantity. If $f(t)$ has period 2ℓ, it has average power

$$P = \tfrac{1}{2\ell}\int_{-\ell}^{\ell}|f(t)|^2\ dt$$

We can express this power in terms of the Fourier coefficients of f just by substituting $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{\ell}t}$

$$P = \tfrac{1}{2\ell}\int_{-\ell}^{\ell} f(t)\overline{f(t)}\ dt = \tfrac{1}{2\ell}\sum_{k,m=-\infty}^{\infty}\int_{-\ell}^{\ell} c_k e^{ik\frac{\pi}{\ell}t}\overline{c_m e^{im\frac{\pi}{\ell}t}}\ dt = \tfrac{1}{2\ell}\sum_{k,m=-\infty}^{\infty} c_k\overline{c_m}\int_{-\ell}^{\ell} e^{i(k-m)\frac{\pi}{\ell}t}\ dt$$

By the orthogonality relation (2), with 17 replaced by m,

$$\int_{-\ell}^{\ell} e^{i(k-m)\frac{\pi}{\ell}t}\ dt = \begin{cases} 2\ell & \text{if } k=m \\ 0 & \text{otherwise}\end{cases}$$

So all of the terms in the double sum with $k \neq m$ are zero and we are left with

$$P = \tfrac{1}{2\ell}\int_{-\ell}^{\ell}|f(t)|^2\ dt = \tfrac{1}{2\ell}\sum_{\substack{k,m=-\infty\\ k=m}}^{\infty} c_k\overline{c_m}\ 2\ell = \sum_{k=-\infty}^{\infty}|c_k|^2 \tag{18}$$

This is called Parseval's relation. To convert it into a statement about the Fourier coefficients a_k, b_k, we just need observe, from (10), that $c_0 = \frac{1}{2}a_0$ and, for $k>0$, $c_k = \frac{1}{2}(a_k - ib_k)$, $c_{-k} = \frac{1}{2}(a_k + ib_k)$. Assuming that a_k and b_k are real, $|c_0|^2 = \frac{1}{4}|a_0|^2$ and, for $k>0$, $|c_k|^2 = |c_{-k}|^2 = \frac{1}{4}[a_k^2 + b_k^2]$ so that

$$P = \tfrac{1}{2\ell}\int_{-\ell}^{\ell}|f(t)|^2\ dt = \tfrac{1}{4}a_0^2 + \tfrac{1}{2}\sum_{k=1}^{\infty}\left[a_k^2 + b_k^2\right]$$

Suppose that we have some complicated signal $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{\ell}t}$ and we wish to approximate it by a simple signal that only contains the finite number of frequencies $k\frac{\pi}{\ell}$ with, for example, $k = -2, -1, 0, 1, 2$.

That is, our approximating signal is to be of the form $F(t) = d_{-2}e^{-2i\frac{\pi}{\ell}t} + d_{-1}e^{-i\frac{\pi}{\ell}t} + d_0 + d_1 e^{i\frac{\pi}{\ell}t} + d_2 e^{2i\frac{\pi}{\ell}t}$. What should we choose for the coefficients d_{-2}, d_{-1}, d_0, d_1 and d_2? To ensure the the error carries minimum average power, we should choose the coefficients so that

$$\tfrac{1}{2\ell}\int_{-\ell}^{\ell} |f(t) - F(t)|^2 \, dt$$

is as small as possible. Since

$$f(t) - F(t) = \Big[\sum_{k=-\infty}^{\infty} c_k e^{ik\frac{\pi}{\ell}t}\Big] - \Big[\sum_{k=-2}^{2} d_k e^{ik\frac{\pi}{\ell}t}\Big] = \sum_{k=-\infty}^{\infty} e_k e^{ik\frac{\pi}{\ell}t}$$

with

$$e_k = \begin{cases} c_k - d_k & \text{if } |k| \le 2 \\ c_k & \text{otherwise} \end{cases}$$

Parseval's relation (18) gives that the average power carried by the error is

$$\tfrac{1}{2\ell}\int_{-\ell}^{\ell} |f(t) - F(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |e_k|^2 = \sum_{-2\le k\le 2} |c_k - d_k|^2 + \sum_{\substack{-\infty<k<\infty \\ |k|>2}} |c_k|^2$$

This is minimised by choosing $d_k = c_k$ for all $-2 \le k \le 2$.

Example 9 By Example 2, the function

$$f(t) = \left\{\begin{matrix} 1 & \text{if } 0 < t < \pi \\ -1 & \text{if } -\pi < t < 0 \end{matrix}\right\} \qquad f(t) = f(t + 2\pi)$$

has Fourier coefficients

$$c_k = \begin{cases} -\frac{2}{k\pi}i & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

So Parseval's relation (18) gives that

$$\tfrac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 \, dt = \sum_{\substack{-\infty<k<\infty \\ k \text{ odd}}} \big| -i\tfrac{2}{k\pi}\big|^2 = \sum_{\substack{-\infty<k<\infty \\ k \text{ odd}}} \tfrac{4}{k^2\pi^2} = \sum_{\substack{0<k<\infty \\ k \text{ odd}}} \tfrac{8}{k^2\pi^2}$$

Since $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 \, dt = 1$, we conclude that

$$\sum_{\substack{0<k<\infty \\ k \text{ odd}}} \tfrac{8}{k^2\pi^2} = 1 \qquad \text{or} \qquad \tfrac{1}{1^2} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \cdots = \tfrac{\pi^2}{8}$$