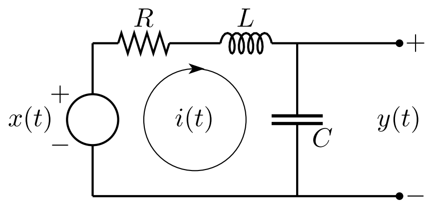
The RLC Circuit

The RLC circuit is the electrical circuit consisting of a resistor of resistance R, a coil of inductance L, a capacitor of capacitance C and a voltage source arranged in series. We're going to think of the voltage $x(t)$



as an input signal and the voltage $y(t)$ as an output signal. The goal is to determine the output voltage for a given input voltage. If $i(t)$ is the current flowing at time t in the loop as shown and $q(t)$ is the charge on the capacitor, then the voltage across R, L and C, respectively, at time t are $Ri(t)$, $L\frac{di}{dt}(t)$ and $y(t) = \frac{q(t)}{C}$. By the Kirchhoff's law that says that the voltage between any two points has to be independent of the path used to travel between the two points, these three voltages must add up to $x(t)$ so that

$$Ri(t) + L\tfrac{di}{dt}(t) + \tfrac{q(t)}{C} = x(t) \tag{1}$$

Assuming that R, L, C and $x(t)$ are known, this is still one differential equation in two unknowns, $i(t)$ and $q(t)$. Fortunately, there is a relationship between the two. Namely

$$i(t) = \tfrac{dq}{dt}(t) = Cy'(t) \tag{2}$$

This just says that the capacitor cannot create or destroy charge on its own. All charging of the capacitor must come from the current. Subbing (2) into (1) gives

$$LCy''(t) + RCy'(t) + y(t) = x(t) \tag{3}$$

For an ac voltage source, choosing the origin of time so that $x(0) = 0$, $x(t) = E_0 \sin(\omega t)$ and the differential equation becomes

$$LCy''(t) + RCy'(t) + y(t) = E_0 \sin(\omega t) \tag{4}$$

One Solution

We first guess one solution of (4) by trying $y_p(t) = A\sin(\omega t - \varphi)$ with the amplitude A and phase φ to be determined. That is, we are guessing that the circuit responds to an oscillating applied voltage with a current that oscillates at the same frequency. For $y_p(t)$ to be a solution, we need

$$\begin{aligned} LCy_p''(t) + RCy_p'(t) + y_p(t) &= E_0 \sin(\omega t) \qquad (4_p) \\ -LC\omega^2 A\sin(\omega t - \varphi) + RC\omega A\cos(\omega t - \varphi) + A\sin(\omega t - \varphi) &= E_0 \sin(\omega t) \\ &= E_0 \sin(\omega t - \varphi + \varphi) \end{aligned}$$

and hence, applying $\sin(A + B) = \sin A \cos B + \cos A \sin B$ with $A = \omega t - \varphi$ and $B = \varphi$,

$$\big(1 - LC\omega^2\big)A\sin(\omega t - \varphi) + RC\omega A\cos(\omega t - \varphi) = E_0 \cos(\varphi)\sin(\omega t - \varphi) + E_0 \sin(\varphi)\cos(\omega t - \varphi)$$

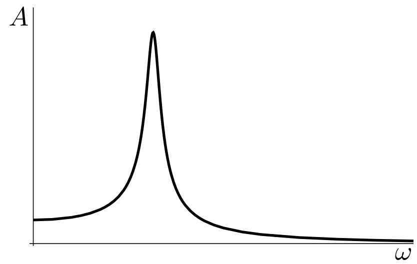
Matching coefficients of $\sin(\omega t - \varphi)$ and $\cos(\omega t - \varphi)$ on the left and right hand sides gives

$$\big(1 - LC\omega^2\big)A = E_0 \cos(\varphi) \tag{5}$$
$$RC\omega A = E_0 \sin(\varphi) \tag{6}$$

It is now easy to solve for A and φ

$$\begin{aligned} \frac{(6)}{(5)} &\Longrightarrow \tan(\varphi) = \frac{RC\omega}{1 - LC\omega^2} && \Longrightarrow \varphi = \tan^{-1}\frac{RC\omega}{1 - LC\omega^2} \\ \sqrt{(5)^2 + (6)^2} &\Longrightarrow \sqrt{\big(1 - LC\omega^2\big)^2 + R^2C^2\omega^2}\ A = E_0 && \Longrightarrow A = \frac{E_0}{\sqrt{(1 - LC\omega^2)^2 + R^2C^2\omega^2}} \end{aligned} \tag{7}$$

Naturally, different input frequencies ω give different output amplitudes A. Here is a graph of A against ω, with all other parameters held fixed.



Note that there is a small range of frequencies that give a large amplitude response. This is the phenomenon of resonance. It is exploited in the design of radio and television tuning circuitry. It has also been dramatically illustrated in, for example, the collapse of the Tacoma narrows bridge.

The General Solution When $R^2 \neq \frac{4L}{C}$

We have found one solution. Now we see if we can find any others. Note that subtracting (4_p) from (4) gives

$$LC(y - y_p)''(t) + RC(y - y_p)'(t) + (y - y_p)(t) = 0$$

That is, a function $y(t)$ is a solution of (4) if and only if $y_c(t) = y(t) - y_p(t)$ is a solution of

$$LCy_c''(t) + RCy_c'(t) + y_c(t) = 0 \tag{$4_{\rm c}$}$$

This is called the complementary homogeneous equation for (4). It is constructed from (4) by deleting all y independent terms. We now guess many solutions to ($4_{\rm c}$). Any solution, $y_c(t)$, of ($4_{\rm c}$) has to have the property that $y_c(t)$, $RCy_c'(t)$ and $LCy_c''(t)$ have to cancel each other out for all t. We choose our guess so that $y_c(t)$, $y_c'(t)$ and $y_c''(t)$ are all proportional to a single function of t. Then it will easy to see if $y_c(t)$, $RCy_c'(t)$ and $LCy_c''(t)$ all cancel. Hence we try $y_c(t) = e^{rt}$, with the constant r to the determined. This guess is a solution of ($4_{\rm c}$) if and only if

$$LCr^2e^{rt} + RCre^{rt} + e^{rt} = 0 \iff LCr^2 + RCr + 1 = 0 \iff r = \tfrac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} \equiv r_{1,2} \tag{8}$$

We now know that $e^{r_1 t}$ and $e^{r_2 t}$ both obey ($4_{\rm c}$). That is,

$$LC\tfrac{d^2}{dt^2}e^{r_1 t} + RC\tfrac{d}{dt}e^{r_1 t} + e^{r_1 t} = 0 \tag{9a}$$
$$LC\tfrac{d^2}{dt^2}e^{r_2 t} + RC\tfrac{d}{dt}e^{r_2 t} + e^{r_2 t} = 0 \tag{9b}$$

for all t. Now let c_1 and c_2 be any constants. Multiplying (9a) by c_1 and (9b) by c_2 and adding gives

$$LC\tfrac{d^2}{dt^2}\big(c_1e^{r_1t}+c_2e^{r_2t}\big)+RC\tfrac{d}{dt}\big(c_1e^{r_1t}+c_2e^{r_2t}\big)+\big(c_1e^{r_1t}+c_2e^{r_2t}\big)=0$$

So $c_1e^{r_1t}+c_2e^{r_2t}$ is also a solution of $(4_{\rm c})$. We now have a two parameter family of solutions of $(4_{\rm c})$. It is reasonable to guess that, to solve a differential equation involving a second derivative, one has to integrate twice so that the general solution contains two arbitrary constants. This is in fact the case, though the proof, or even giving a precise meaning to "the general solution contains two arbitrary constants", is beyond the scope of this course. Assuming that $R^2\ne\frac{4L}{C}$, r_1 and r_2 are different and the general solution to (4_c) is $c_1e^{r_1t}+c_2e^{r_2t}$. Then, the general solution of (4) is

$$y(t) \ = \ c_1e^{r_1t}+c_2e^{r_2t}+A\sin(\omega t-\varphi)$$

with $r_1,\ r_2$ given in (8) and $A,\ \varphi$ given in (7).

The arbitrary constants c_1 and c_2 are to be determined by initial conditions. However, when $R,L,C>0$, as is often the case, e^{r_1t} and e^{r_2t} damp out exponentially and their values are not very important. To see this, observe that, as we are assuming $R^2\ne\frac{4L}{C}$, R^2C^2-4LC is either strictly positive or negative.

- In the event that it is positive $0<R^2C^2-4LC<R^2C^2$, so that $0<\sqrt{R^2C^2-4LC}<RC$ and $-RC\pm\sqrt{R^2C^2-4LC}<0$. So, when $R^2C^2-4LC>0$, both r_1 and r_2 are negative and both e^{r_1t} and e^{r_2t} decay exponentially to zero as $t\to\infty$.
- In the event that R^2C^2-4LC is negative, $\sqrt{R^2C^2-4LC}$ is pure imaginary. Write $\frac{R}{2L}=\rho$ and $\frac{1}{2LC}\sqrt{R^2C^2-4LC}=i\omega$. Then

$$\begin{aligned}c_1e^{r_1t}+c_2e^{r_2t}&=c_1e^{(-\rho+i\omega)t}+c_2e^{(-\rho-i\omega)t}=e^{-\rho t}\big[c_1e^{i\omega t}+c_2e^{-i\omega t}\big]\\&=e^{-\rho t}\big[c_1\big(\cos(\omega t)+i\sin(\omega t)\big)+c_2\big(\cos(\omega t)-i\sin(\omega t)\big)\big]\\&=e^{-\rho t}\big[d_1\cos(\omega t)+d_2\sin(\omega t)\big]\end{aligned}$$

  where $d_1=c_1+c_2$ and $d_2=ic_1-ic_2$. The factor $\big[d_1\cos(\omega t)+d_2\sin(\omega t)\big]$ just oscillates in time. It remains bounded. On the other hand $e^{-\rho t}$ decays exponentially. So, for any c_1,c_2, $c_1e^{r_1t}+c_2e^{r_2t}$ decays exponentially to zero as $t\to\infty$.

The General Solution When $R^2=\frac{4L}{C}$

From now on, assume $R^2=\frac{4L}{C}$. To save writing, set $\rho=\frac{R}{2L}$. Now, $r_1=r_2=\rho$, so

$$c_1e^{r_1t}+c_2e^{r_2t}=(c_1+c_2)e^{-\rho t}=de^{-\rho t}$$

where $d=c_1+c_2$. We really only have a one parameter family of solutions to $(4_{\rm c})$. The general solution to $(4_{\rm c})$ contains two independent free parameters. Here is a trick (called reduction of order) for finding the other solutions: look for solutions of the form $v(t)e^{-\rho t}$. Here $e^{-\rho t}$ is the solution we have already found and $v(t)$ is to be determined. To save writing, divide $(4_{\rm c})$ by LC and substitute that $\frac{R}{L}=2\rho$ and $\frac{1}{LC}=\frac{R^2}{4L^2}=\rho^2$ (recall that we are assuming that $R^2=\frac{4L}{C}$). So $(4_{\rm c})$ is equivalent to

$$y_c''(t)+2\rho\,y_c'(t)+\rho^2\,y_c(t)=0$$

Sub in

$$\begin{aligned}y_c(t)&=\qquad v(t)e^{-\rho t}\\y_c'(t)&=-\rho\ v(t)e^{-\rho t}+\quad v'(t)e^{-\rho t}\\y_c''(t)&=\ \rho^2v(t)e^{-\rho t}-2\rho v'(t)e^{-\rho t}+v''(t)e^{-\rho t}\end{aligned}$$

When $y_c(t) = v(t)e^{-\rho t}$,

$$\begin{aligned} y_c''(t) + 2\rho\, y_c'(t) + \rho^2\, y_c(t) &= \big[\rho^2 - 2\rho^2 + \rho^2\big] v(t)e^{-\rho t} + \big[-2\rho + 2\rho\big] v'(t)e^{-\rho t} + v''(t)e^{-\rho t} \\ &= v''(t)e^{-\rho t} \end{aligned}$$

Thus $v(t)e^{-\rho t}$ is a solution of $(4_{\rm c})$ whenever the function $v''(t) = 0$ for all t. But, for any values of the constants c_1 and c_2, $v(t) = c_1 + c_2 t$ has vanishing second derivative so $\big(c_1 + c_2 t\big)e^{-\rho t}$ solves $(4_{\rm c})$ and

$$y(t) \;=\; \big(c_1 + c_2 t\big)e^{-\rho t} + A\sin(\omega t - \varphi)$$

solves (4). This is the general solution.