

Properties of Vector Operations

Addition and Scalar Multiplication

1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
3. $\vec{a} + \vec{0} = \vec{a}$
4. $\vec{a} + (-\vec{a}) = \vec{0}$
5. $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
6. $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
7. $(cd)\vec{a} = c(d\vec{a})$
8. $1\vec{a} = \vec{a}$

Dot Product

The dot product is defined by

$$\vec{a} = \langle a_1, a_2, a_3 \rangle, \vec{b} = \langle b_1, b_2, b_3 \rangle$$
$$\implies \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and obeys

0. \vec{a}, \vec{b} are vectors and $\vec{a} \cdot \vec{b}$ is a number
1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b})$
5. $\vec{0} \cdot \vec{a} = 0$
6. $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$
7. $\vec{a} \cdot \vec{b} = 0 \iff \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a} \perp \vec{b}$

In property 6, θ is the angle between \vec{a} and \vec{b} .

Cross Product

The cross product is defined by

$$\vec{a} = \langle a_1, a_2, a_3 \rangle, \quad \vec{b} = \langle b_1, b_2, b_3 \rangle$$
$$\implies \vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

and obeys

0. \vec{a}, \vec{b} and $\vec{a} \times \vec{b}$ are all vectors in three dimensions
1. $\vec{a} \times \vec{b} \perp \vec{a}, \vec{b}$
2. $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$
3. $\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$
4. $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$
5. $\vec{a} \times \vec{b} = \vec{0} \iff \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0} \text{ or } \vec{a} \parallel \vec{b}$
6. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
7. $(c\vec{a}) \times \vec{b} = \vec{a} \times (c\vec{b}) = c(\vec{a} \times \vec{b})$
8. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
9. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
10. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{b} \cdot \vec{a})\vec{c}$

In properties 2 and 4, θ is the angle between \vec{a} and \vec{b} .

In property 4, $|\hat{n}| = 1$, $\hat{n} \perp \vec{a}, \vec{b}$ and $(\vec{a}, \vec{b}, \hat{n})$ obey the right hand rule.

WARNING: Take particular care with properties 6 and 10. They are counterintuitive and cause huge numbers of errors. In particular,

$$\begin{aligned}\vec{a} \times \vec{b} &\neq \vec{b} \times \vec{a} \\ \vec{a} \times (\vec{b} \times \vec{c}) &\neq (\vec{a} \times \vec{b}) \times \vec{c}\end{aligned}$$

for most \vec{a} , \vec{b} and \vec{c} . For example

$$\begin{aligned}\hat{i} \times (\hat{i} \times \hat{j}) &= \hat{i} \times \hat{k} = -\hat{k} \times \hat{i} = -\hat{j} \\ (\hat{i} \times \hat{i}) \times \hat{j} &= \vec{0} \times \hat{j} = \vec{0}\end{aligned}$$