Example of the Use of Stokes’ Theorem

In these notes we compute, in three different ways, \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} = (z-y) \mathbf{i} - (x+z) \mathbf{j} - (x+y) \mathbf{k} \) and \( C \) is the curve \( x^2 + y^2 + z^2 = 4 \), \( z = y \) oriented counterclockwise when viewed from above.

**Direct Computation**

In this first computation, we parametrize the curve \( C \) and compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \) directly. The plane \( z = y \) passes through the origin, which is the centre of the sphere \( x^2 + y^2 + z^2 = 4 \). So \( C \) is a circle which, like the sphere, has radius 2 and centre \((0,0,0)\). We use a parametrization of the form

\[
\mathbf{r}(t) = \mathbf{c} + \rho \cos t \mathbf{i}' + \rho \sin t \mathbf{j}' \\
0 \leq t \leq 2\pi
\]

where \( \mathbf{c} = (0,0,0) \) is the centre of \( C \), \( \rho = 2 \) is the radius of \( C \) and \( \mathbf{i}' \) and \( \mathbf{j}' \) are two vectors that (a) are unit vectors, (b) are parallel to the plane \( z = y \) and (c) are mutually perpendicular. The point \((2,0,0)\) satisfies both \( x^2 + y^2 + z^2 = 4 \) and \( z = y \) and so is on \( C \). We may choose \( \mathbf{i}' \) to be the unit vector in the direction from the centre \((0,0,0)\) of the circle towards \((2,0,0)\). Namely \( \mathbf{i}' = (1,0,0) \). Since the plane of the circle is \( z - y = 0 \), the vector \( \mathbf{V}'(z-y) = (0,1,1) \) is perpendicular to the plane of \( C \). So \( \mathbf{k}' = \frac{1}{\sqrt{2}}(0,-1,1) \) is a unit vector normal to \( z = y \). Then \( \mathbf{j}' = \mathbf{k}' \times \mathbf{i}' = \frac{1}{\sqrt{2}}(0,1,1) \times (1,0,0) = \frac{1}{\sqrt{2}}(0,1,1) \) is a unit vector that is perpendicular to \( \mathbf{i}' \). Since \( \mathbf{j}' \) is also perpendicular to \( \mathbf{k}' \), it is parallel to \( z = y \). Subbing in \( \mathbf{c} = (0,0,0) \), \( \rho = 2 \), \( \mathbf{i}' = (1,0,0) \) and \( \mathbf{j}' = \frac{1}{\sqrt{2}}(0,1,1) \) gives

\[
\mathbf{r}(t) = 2 \cos t (1,0,0) + 2 \sin t \frac{1}{\sqrt{2}}(0,1,1) = 2 \left( \cos t, \frac{\sin t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}} \right) \\
0 \leq t \leq 2\pi
\]

To check that this parametrization is correct, note that \( x = 2 \cos t \), \( y = \sqrt{2} \sin t \), \( z = \sqrt{2} \sin t \) satisfies both \( x^2 + y^2 + z^2 = 4 \) and \( z = y \). At \( t = 0 \), \( \mathbf{r}(0) = (2,0,0) \). As \( t \) increases, \( z(t) = \sqrt{2} \) increases and \( \mathbf{r}(t) \) moves upwards towards \( \mathbf{r}(\frac{\pi}{2}) = (0,\sqrt{2},\sqrt{2}) \). This is the desired counterclockwise direction. Now that we have a parametrization, we can set up the integral.

\[
\begin{align*}
\mathbf{r}(t) &= (2 \cos t, \sqrt{2} \sin t, \sqrt{2} \sin t) \\
\mathbf{r}'(t) &= (-2 \sin t, \sqrt{2} \cos t, \sqrt{2} \cos t) \\
\mathbf{F}(\mathbf{r}(t)) &= (z(t) - y(t), -x(t) - z(t), -x(t) - y(t)) \\
&= (\sqrt{2} \sin t - \sqrt{2} \sin t, -2 \cos t - \sqrt{2} \sin t, -2 \cos t - \sqrt{2} \sin t) \\
&= (-2 \cos t + \sqrt{2} \sin t, 2 \cos t + \sqrt{2} \sin t) \\
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= -[\sqrt{2} \cos^2 t + 4 \cos t \sin t] = -[2 \sqrt{2} \cos(2t) + 2 \sqrt{2} + 2 \sin(2t)] \\
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\
&= \int_0^{2\pi} \left[ -[2 \sqrt{2} \cos(2t) + 2 \sqrt{2} + 2 \sin(2t)] \right] \, dt = -[\sqrt{2} \sin(2t) + 2 \sqrt{2} - \cos(2t)]_0^{2\pi} = -4 \sqrt{2} \pi
\end{align*}
\]
Stokes’ Theorem

To apply Stokes’ theorem we need to express $C$ as the boundary $\partial S$ of a surface $S$. As

$$C = \{(x, y, z) \mid x^2 + y^2 + z^2 = 4, \ z = y \}$$

is a closed curve, this is possible. In fact there are many possible choices of $S$ with $\partial S = C$. Three possible $S$’s are

$$S = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 4, \ z = y \}$$

$$S' = \{(x, y, z) \mid x^2 + y^2 + z^2 = 4, \ z \geq y \}$$

$$S'' = \{(x, y, z) \mid x^2 + y^2 + z^2 = 4, \ z \leq y \}$$

The first of these, which is part of a plane, is likely to lead to simpler computations than the last two, which are parts of a sphere. So we choose to use it.

In preparation for application of Stokes’ theorem, we compute $\vec{\nabla} \times \vec{F}$ and $\hat{n} \, dS$. For the latter, we apply the formula $\hat{n} \, dS = \pm (-f_x, -f_y, 1) \, dxdy$ to the surface $z = f(x, y) = y$. We use the + sign to give the normal a positive $\hat{k}$ component.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
z - y & -x - z & -x - y
\end{vmatrix} = \hat{i}(1) - \hat{j}(1) + \hat{k}(1) = 2\hat{j}$$

$$\hat{n} \, dS = (0, -1, 1) \, dxdy$$

$$\vec{\nabla} \times \vec{F} \cdot \hat{n} \, dS = (0, 2, 0) \cdot (0, -1, 1) \, dxdy = -2 \, dxdy$$

The integration variables are $x$ and $y$ and, by definition, the domain of integration is

$$R = \{(x, y) \mid (x, y, z) \text{ is in } S \text{ for some } z \}$$

To determine precisely what this domain of integration is, we observe that since $z = y$ on $S$

$$S = \{(x, y, z) \mid x^2 + 2y^2 \leq 4, \ z = y \} \implies R = \{(x, y) \mid x^2 + 2y^2 \leq 4 \}$$

So the domain of integration is an ellipse with semimajor axis $a = 2$, semiminor axis $b = \sqrt{2}$ and area $\pi ab = 2\sqrt{2}\pi$ and

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n} \, dS = \iiint_R (-2) \, dxdy = -2 \text{ Area } (R) = -4\sqrt{2}\pi$$
Remark (Limits of integration) If the integrand were more complicated, we would have to evaluate the integral over $R$ by expressing it as an iterated integrals with the correct limits of integration. First suppose that we slice up $R$ using thin vertical slices. On each such slice, $x$ is essentially constant and $y$ runs from $-\sqrt{4 - x^2}/2$ to $\sqrt{4 - x^2}/2$. The leftmost such slice would have $x = -2$ and the rightmost such slice would have $x = 2$. So the correct limits with this slicing are

\[
\int_{-2}^{2} dy \int_{-\sqrt{4 - x^2}/2}^{\sqrt{4 - x^2}/2} f(x, y) dx
\]

If, instead, we slice up $R$ using thin horizontal slices, then, on each such slice, $y$ is essentially constant and $x$ runs from $-\sqrt{4 - 2y^2}$ to $\sqrt{4 - 2y^2}$. The bottom such slice would have $y = -\sqrt{2}$ and the top such slice would have $y = \sqrt{2}$. So the correct limits with this slicing are

\[
\int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{-\sqrt{4 - 2y^2}}^{\sqrt{4 - 2y^2}} f(x, y) dy
\]

Note that the integral with limits

\[
\int_{-\sqrt{2}}^{\sqrt{2}} dy \int_{-2}^{2} dx f(x, y)
\]

corresponds to a slicing with $x$ running from $-2$ to $2$ on every slice. This corresponds to a rectangular domain of integration.

Stokes’ Theorem, Again

Since the integrand is just a constant and $S$ is so simple, we can evaluate the integral $\iint_S \nabla \times \vec{F} \cdot \hat{n} \ dS$ without ever determining $dS$ explicitly and without ever setting up any limits of integration. We already know that $\nabla \times \vec{F} = 2\hat{j}$. Since $S$ is the level surface $z - y = 0$, the gradient $\nabla(z - y) = -\hat{j} + \hat{k}$ is normal to $S$. So $\hat{n} = \frac{1}{\sqrt{2}}(-\hat{j} + \hat{k})$ and

\[
\oint_C \vec{F} \cdot d\vec{r} = \int_S \nabla \times \vec{F} \cdot \hat{n} \ dS = \int_S (2\hat{j}) \cdot \frac{1}{\sqrt{2}}(-\hat{j} + \hat{k}) \ dS = \int_S -\sqrt{2} \ dS = -\sqrt{2} \ \text{Area} \ (S)
\]

As $S$ is a circle of radius $2$, $\oint_C \vec{F} \cdot d\vec{r} = -4\sqrt{2}\pi$, yet again.