Equality of Mixed Partials

Theorem. *If the partial derivatives* $\frac{\partial^2 f}{\partial x \partial y}$ *and* $\frac{\partial^2 f}{\partial y \partial x}$ *exist and are continuous at* (x_0, y_0)*, then*

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

Proof: Here is an outline of the proof. The details are given as footnotes at the end of the outline. Fix x_0 and y_0 and define$^{(1)}$

$$F(h,k) = \tfrac{1}{hk}\big[f(x_0+h, y_0+k) - f(x_0, y_0+k) - f(x_0+h, y_0) + f(x_0, y_0)\big]$$

Then, by the mean value theorem,

$$\begin{aligned}
F(h,k) &\overset{2}{=} \tfrac{1}{h}\Big[\tfrac{\partial f}{\partial y}(x_0+h, y_0+\theta_1 k) - \tfrac{\partial f}{\partial y}(x_0, y_0+\theta_1 k)\Big] \\
&\overset{3}{=} \tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial y}(x_0+\theta_2 h, y_0+\theta_1 k) \\
F(h,k) &\overset{4}{=} \tfrac{1}{k}\Big[\tfrac{\partial f}{\partial x}(x_0+\theta_3 h, y_0+k) - \tfrac{\partial f}{\partial x}(x_0+\theta_3 h, y_0)\Big] \\
&\overset{5}{=} \tfrac{\partial}{\partial y}\tfrac{\partial f}{\partial x}(x_0+\theta_3 h, y_0+\theta_4 k)
\end{aligned}$$

for some $0 < \theta_1, \theta_2, \theta_3, \theta_4 < 1$. All of $\theta_1, \theta_2, \theta_3, \theta_4$ depend on x_0, y_0, h, k. Hence

$$\tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial y}(x_0+\theta_2 h, y_0+\theta_1 k) = \tfrac{\partial}{\partial y}\tfrac{\partial f}{\partial x}(x_0+\theta_3 h, y_0+\theta_4 k)$$

for all h and k. Taking the limit $(h,k) \to (0,0)$ and using the assumed continuity of both partial derivatives at (x_0, y_0) gives

$$\tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial y}(x_0, y_0) = \tfrac{\partial}{\partial y}\tfrac{\partial f}{\partial x}(x_0, y_0)$$

The Details

(1) We define $F(h,k)$ in this way because both partial derivatives $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$ and $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$ are defined as limits of $F(h,k)$ as $h, k \to 0$. For example,

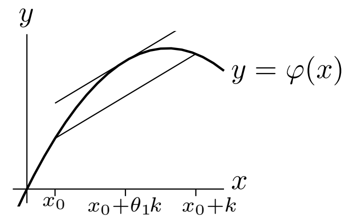
$$\begin{aligned}
\tfrac{\partial}{\partial y}\tfrac{\partial f}{\partial x}(x_0, y_0) &= \lim_{k\to 0} \tfrac{1}{k}\Big[\tfrac{\partial f}{\partial x}(x_0, y_0+k) - \tfrac{\partial f}{\partial x}(x_0, y_0)\Big] \\
&= \lim_{k\to 0} \tfrac{1}{k}\Big[\lim_{h\to 0} \tfrac{f(x_0+h, y_0+k) - f(x_0, y_0+k)}{h} - \lim_{h\to 0} \tfrac{f(x_0+h, y_0) - f(x_0, y_0)}{h}\Big] \\
&= \lim_{k\to 0}\lim_{h\to 0} \tfrac{f(x_0+h, y_0+k) - f(x_0, y_0+k) - f(x_0+h, y_0) + f(x_0, y_0)}{hk} \\
&= \lim_{k\to 0}\lim_{h\to 0} F(h,k)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial y}(x_0, y_0) &= \lim_{h\to 0} \tfrac{1}{h}\Big[\tfrac{\partial f}{\partial y}(x_0+h, y_0) - \tfrac{\partial f}{\partial y}(x_0, y_0)\Big] \\
&= \lim_{h\to 0} \tfrac{1}{h}\Big[\lim_{k\to 0} \tfrac{f(x_0+h, y_0+k) - f(x_0+h, y_0)}{k} - \lim_{k\to 0} \tfrac{f(x_0, y_0+k) - f(x_0, y_0)}{k}\Big] \\
&= \lim_{h\to 0}\lim_{k\to 0} \tfrac{f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x_0, y_0+k) + f(x_0, y_0)}{hk} \\
&= \lim_{h\to 0}\lim_{k\to 0} F(h,k)
\end{aligned}$$

(2) The mean value theorem says that, for any differentiable function $\varphi(x)$, the slope of the line joining the points $\big(x_0, \varphi(x_0)\big)$ and $\big(x_0 + k, \varphi(x_0 + k)\big)$ on the graph of φ is the same as the slope of the tangent to the graph at some point between x_0 and $x_0 + k$. This is, there is some $0 < \theta_1 < 1$ such that

$$\tfrac{\varphi(x_0+k)-\varphi(x_0)}{k} = \tfrac{d\varphi}{dx}(x_0 + \theta_1 k)$$



Applying this with x replaced by y and φ replaced by $G(y) = f(x_0 + h, y) - f(x_0, y)$ gives

$$\begin{aligned} \tfrac{G(y_0+k)-G(y_0)}{k} &= \tfrac{dG}{dy}(y_0 + \theta_1 k) \qquad \text{for some } 0 < \theta_1 < 1 \\ &= \tfrac{\partial f}{\partial y}(x_0 + h, y_0 + \theta_1 k) - \tfrac{\partial f}{\partial y}(x_0, y_0 + \theta_1 k) \end{aligned}$$

Hence, for some $0 < \theta_1 < 1$,

$$F(h,k) \;=\; \tfrac{1}{h}\Big[\tfrac{G(y_0+k)-G(y_0)}{k}\Big] = \tfrac{1}{h}\Big[\tfrac{\partial f}{\partial y}(x_0 + h, y_0 + \theta_1 k) - \tfrac{\partial f}{\partial y}(x_0, y_0 + \theta_1 k)\Big]$$

(3) Define $H(x) = \frac{\partial f}{\partial y}(x, y_0 + \theta_1 k)$. By the mean value theorem

$$\begin{aligned} F(h,k) &= \tfrac{1}{h}\Big[H(x_0 + h) - H(x_0)\Big] \\ &= \tfrac{dH}{dx}(x_0 + \theta_2 h) \qquad \text{for some } 0 < \theta_2 < 1 \\ &= \tfrac{\partial}{\partial x}\tfrac{\partial f}{\partial y}(x_0 + \theta_2 h, y_0 + \theta_1 k) \end{aligned}$$

(4) Define $A(x) = f(x, y_0 + k) - f(x, y_0)$. By the mean value theorem

$$\begin{aligned} F(h,k) &= \tfrac{1}{k}\Big[\tfrac{A(x_0+h)-A(x_0)}{h}\Big] \\ &= \tfrac{1}{k}\tfrac{dA}{dx}(x_0 + \theta_3 h) \qquad \text{for some } 0 < \theta_3 < 1 \\ &= \tfrac{1}{k}\Big[\tfrac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0 + k) - \tfrac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0)\Big] \end{aligned}$$

(5) Define $B(y) = \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y)$. By the mean value theorem

$$\begin{aligned} F(h,k) &= \tfrac{1}{k}\Big[B(y_0 + k) - B(y_0)\Big] \\ &= \tfrac{dB}{dy}(y_0 + \theta_4 k) \qquad \text{for some } 0 < \theta_4 < 1 \\ &= \tfrac{\partial}{\partial y}\tfrac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0 + \theta_4 k) \end{aligned}$$

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