Errors in Measurement

The Question

Suppose that three variables are measured with percentage error at most ε_1, ε_2 and ε_3 respectively. In other words, if the exact value of variable number i is x_i and measured value of variable number i is $x_i + \Delta x_i$ then

$$100\left|\tfrac{\Delta x_i}{x_i}\right| \le \varepsilon_i$$

Suppose further that a quantity P is then computed by taking the product of the three variables. For example, if x_1, x_2 and x_3 are the lengths of the three edges of a rectangular box, then P is the volume of the box. The question is "What is the percentage error in this measured value of P?".

The answer

The exact value of P is

$$P(x_1, x_2, x_3) = x_1 x_2 x_3$$

and the measured value of P is $P(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3)$. So, the percentage error in $P(x_1, x_2, x_3)$ is

$$100\left|\frac{P(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) - P(x_1, x_2, x_3)}{P(x_1, x_2, x_3)}\right|$$

We can get a much simpler approximate expression for this percentage error, which is good enough for virtually all applications, by applying

$$\begin{aligned} &P(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) \\ &\qquad \approx P(x_1, x_2, x_3) + P_{x_1}(x_1, x_2, x_3)\Delta x_1 + P_{x_2}(x_1, x_2, x_3)\Delta x_2 + P_{x_3}(x_1, x_2, x_3)\Delta x_3 \end{aligned}$$

The three partial derivatives are

$$\begin{aligned} P_{x_1}(x_1, x_2, x_3) &= \tfrac{\partial}{\partial x_1}[x_1 x_2 x_3] = x_2 x_3 \\ P_{x_2}(x_1, x_2, x_3) &= \tfrac{\partial}{\partial x_2}[x_1 x_2 x_3] = x_1 x_3 \\ P_{x_3}(x_1, x_2, x_3) &= \tfrac{\partial}{\partial x_3}[x_1 x_2 x_3] = x_1 x_2 \end{aligned}$$

So

$$P(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) \approx P(x_1, x_2, x_3) + x_2 x_3 \Delta x_1 + x_1 x_3 \Delta x_2 + x_1 x_2 \Delta x_3$$

and the percentage error in P is

$$\begin{aligned} &100\left|\frac{P(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) - P(x_1, x_2, x_3)}{P(x_1, x_2, x_3)}\right| \\ &\approx 100\left|\frac{x_2 x_3 \Delta x_1 + x_1 x_3 \Delta x_2 + x_1 x_2 \Delta x_3}{P(x_1, x_2, x_3)}\right| \\ &= 100\left|\frac{x_2 x_3 \Delta x_1 + x_1 x_3 \Delta x_2 + x_1 x_2 \Delta x_3}{x_1 x_2 x_3}\right| \\ &= \left|100\frac{\Delta x_1}{x_1} + 100\frac{\Delta x_2}{x_2} + 100\frac{\Delta x_3}{x_3}\right| \\ &\le \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \end{aligned}$$

More generally, if we take a product of n, rather than three, variables the percentage error in the product becomes (approximately) $\sum\limits_{i=1}^{n} \varepsilon_i$. This is the basis of the experamentalist's rule of thumb that when you take products, percentage errors add.