Errors in Measurement

The Question

Suppose that three variables are measured with percentage error at most \( \varepsilon_1 \), \( \varepsilon_2 \) and \( \varepsilon_3 \) respectively. In other words, if the exact value of variable number \( i \) is \( x_i \) and measured value of variable number \( i \) is \( x_i + \Delta x_i \) then

\[
100 \left| \frac{\Delta x_i}{x_i} \right| \leq \varepsilon_i
\]

Suppose further that a quantity \( P \) is then computed by taking the product of the three variables. For example, if \( x_1 \), \( x_2 \) and \( x_3 \) are the lengths of the three edges of a rectangular box, then \( P \) is the volume of the box. The question is “What is the percentage error in this measured value of \( P \)?”

The answer

The exact value of \( P \) is

\[
P(x_1, x_2, x_3) = x_1 x_2 x_3
\]

and the measured value of \( P \) is \( P(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) \). So, the percentage error in \( P(x_1, x_2, x_3) \) is

\[
100 \left| \frac{P(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) - P(x_1, x_2, x_3)}{P(x_1, x_2, x_3)} \right|
\]

We can get a much simpler approximate expression for this percentage error, which is good enough for virtually all applications, by applying

\[
P(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) \approx P(x_1, x_2, x_3) + P_{x_1}(x_1, x_2, x_3) \Delta x_1 + P_{x_2}(x_1, x_2, x_3) \Delta x_2 + P_{x_3}(x_1, x_2, x_3) \Delta x_3
\]

The three partial derivatives are

\[
P_{x_1}(x_1, x_2, x_3) = \frac{\partial}{\partial x_1}[x_1 x_2 x_3] = x_2 x_3
\]

\[
P_{x_2}(x_1, x_2, x_3) = \frac{\partial}{\partial x_2}[x_1 x_2 x_3] = x_1 x_3
\]

\[
P_{x_3}(x_1, x_2, x_3) = \frac{\partial}{\partial x_3}[x_1 x_2 x_3] = x_1 x_2
\]

So

\[
P(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) \approx P(x_1, x_2, x_3) + x_2 x_3 \Delta x_1 + x_1 x_3 \Delta x_2 + x_1 x_2 \Delta x_3
\]

and the percentage error in \( P \) is

\[
100 \left| \frac{P(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) - P(x_1, x_2, x_3)}{P(x_1, x_2, x_3)} \right|
\]

\[
\approx 100 \left| \frac{x_2 x_3 \Delta x_1 + x_1 x_3 \Delta x_2 + x_1 x_2 \Delta x_3}{P(x_1, x_2, x_3)} \right|
\]

\[
= 100 \left| \frac{x_2 x_3 \Delta x_1 + x_1 x_3 \Delta x_2 + x_1 x_2 \Delta x_3}{x_1 x_2 x_3} \right|
\]

\[
= \left| \frac{100 \Delta x_1}{x_1} + \frac{100 \Delta x_2}{x_2} + \frac{100 \Delta x_3}{x_3} \right|
\]

\[
\leq \varepsilon_1 + \varepsilon_2 + \varepsilon_3
\]

More generally, if we take a product of \( n \), rather than three, variables the percentage error in the product becomes (approximately) \( \sum_{i=1}^{n} \varepsilon_i \). This is the basis of the experimentalist’s rule of thumb that when you take products, percentage errors add.