Formulae for dS and $\hat{n}\,dS$

Main Formulae

For the **parametrized surface** $\vec{r}(u,v)$

$$\hat{n}\,dS = \pm\frac{\partial\vec{r}}{\partial u}(u,v)\times\frac{\partial\vec{r}}{\partial v}(u,v)\;dudv$$
$$dS = \left|\frac{\partial\vec{r}}{\partial u}(u,v)\times\frac{\partial\vec{r}}{\partial v}(u,v)\right|\;dudv$$

For the **graph** $z = f(x,y)$

$$\hat{n}\,dS = \pm\big[-f_x(x,y)\hat{\boldsymbol{\imath}} - f_y(x,y)\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big]\;dxdy$$
$$dS = \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\;dxdy$$

For the **level surface** $F(x,y,z) = 0$

$$\hat{n}\,dS = \pm\frac{\vec{\boldsymbol{\nabla}}F(x,y,z(x,y))}{\vec{\boldsymbol{\nabla}}F(x,y,z(x,y))\cdot\hat{\mathbf{k}}}\;dxdy$$
$$dS = \left|\frac{\vec{\boldsymbol{\nabla}}F(x,y,z(x,y))}{\vec{\boldsymbol{\nabla}}F(x,y,z(x,y))\cdot\hat{\mathbf{k}}}\right|\;dxdy$$

Variations

For the surface $x = g(y,z)$

$$\hat{n}\, dS = \pm\big[\hat{\boldsymbol{\imath}} - g_y(y,z)\hat{\boldsymbol{\jmath}} - g_z(y,z)\hat{\mathbf{k}}\big]\; dydz$$

$$dS = \sqrt{1 + g_y(y,z)^2 + g_z(y,z)^2}\; dydz$$

For the surface $y = h(x,z)$

$$\hat{n}\, dS = \pm\big[-h_x(x,z)\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - h_z(x,z)\hat{\mathbf{k}}\big]\; dxdz$$

$$dS = \sqrt{1 + h_x(x,z)^2 + h_z(x,z)^2}\; dxdz$$

For the **level surface** $F(x,y,z) = 0$

$$\begin{aligned}
\hat{n}\, dS &= \pm\frac{\vec{\boldsymbol{\nabla}} F(x,y,z(x,y))}{\vec{\boldsymbol{\nabla}} F(x,y,z(x,y))\cdot\hat{\mathbf{k}}}\; dxdy \\
&= \pm\frac{\vec{\boldsymbol{\nabla}} F(x,y(x,z),z)}{\vec{\boldsymbol{\nabla}} F(x,y(x,z),z)\cdot\hat{\boldsymbol{\jmath}}}\; dxdz \\
&= \pm\frac{\vec{\boldsymbol{\nabla}} F(x(y,z),y,z)}{\vec{\boldsymbol{\nabla}} F(x(y,z),y,z)\cdot\hat{\boldsymbol{\imath}}}\; dydz \\
dS &= \left|\frac{\vec{\boldsymbol{\nabla}} F(x,y,z(x,y))}{\vec{\boldsymbol{\nabla}} F(x,y,z(x,y))\cdot\hat{\mathbf{k}}}\right|\; dxdy \\
&= \left|\frac{\vec{\boldsymbol{\nabla}} F(x,y(x,z),z)}{\vec{\boldsymbol{\nabla}} F(x,y(x,z),z)\cdot\hat{\boldsymbol{\jmath}}}\right|\; dxdz \\
&= \left|\frac{\vec{\boldsymbol{\nabla}} F(x(y,z),y,z)}{\vec{\boldsymbol{\nabla}} F(x(y,z),y,z)\cdot\hat{\boldsymbol{\imath}}}\right|\; dydz
\end{aligned}$$