Directional Derivatives

The Question

Suppose that you leave the point (a,b) moving with velocity $\vec{v} = \langle v_1, v_2\rangle$. Suppose further that the temperature at (x,y) is $f(x,y)$. Then what rate of change of temperature do you feel?

The Answers

Let's set the beginning of time, $t = 0$, to the time at which you leave (a,b). Then at time t you are at $(a+v_1t, b+v_2t)$ and feel the temperature $f(a+v_1t, b+v_2t)$. So the change in temperature between time 0 and time t is $f(a+v_1t, b+v_2t) - f(a,b)$, the average rate of change of temperature, per unit time, between time 0 and time t is $\frac{f(a+v_1t,b+v_2t)-f(a,b)}{t}$ and the instantaneous rate of change of temperature per unit time as you leave (a,b) is $\lim_{t\to 0} \frac{f(a+v_1t,b+v_2t)-f(a,b)}{t}$. We apply the approximation

$$f(a+\Delta x, b+\Delta y) - f(a,b) \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

with $\Delta x = v_1 t$ and $\Delta y = v_2 t$. In the limit as $t \to 0$, the approximation becomes exact and we have

$$\begin{aligned}
\lim_{t\to 0} \tfrac{f(a+v_1t,b+v_2t)-f(a,b)}{t} &= \lim_{t\to 0} \tfrac{f_x(a,b)\,v_1t+f_y(a,b)\,v_2t}{t} \\
&= f_x(a,b)\,v_1 + f_y(a,b)\,v_2 \\
&= \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle v_1, v_2\rangle
\end{aligned}$$

The vector $\langle f_x(a,b), f_y(a,b)\rangle$ is denoted $\vec{\nabla} f(a,b)$ and is called "the **gradient** of the function f at the point (a,b)". It has one component for each variable of f. The $j^{\rm th}$ component is the partial derivative of f with respect to the $j^{\rm th}$ variable, evaluated at (a,b). The expression $\langle f_x(a,b), f_y(a,b)\rangle \cdot \langle v_1, v_2\rangle = \vec{\nabla} f(a,b)\cdot\vec{v}$ is often denoted $D_{\vec{v}} f(a,b)$. So we conclude that *the rate of change of f per unit time as we leave (a,b) moving with velocity $\vec{v}$* is

$$\boxed{D_{\vec{v}} f(a,b) = \vec{\nabla} f(a,b)\cdot\vec{v}}$$

We can compute the rate of change of temperature per unit distance in a similar way. The change in temperature between time 0 and time t is $f(a+v_1t, b+v_2t) - f(a,b)$. Between time 0 and time t you have travelled a distance $|\vec{v}|t$. So the instantaneous rate of change of temperature per unit distance as you leave (a,b) is

$$\lim_{t\to 0} \tfrac{f(a+v_1t,b+v_2t)-f(a,b)}{t|\vec{v}|}$$

This is exactly $\frac{1}{|\vec{v}|}$ times $\lim_{t\to 0} \frac{f(a+v_1t,b+v_2t)-f(a,b)}{t}$ which we computed above to be $D_{\vec{v}} f(a,b)$. So *the rate of change of f per unit distance as we leave (a,b) moving in direction $\vec{v}$* is

$$\boxed{\vec{\nabla} f(a,b)\cdot\tfrac{\vec{v}}{|\vec{v}|} = D_{\frac{\vec{v}}{|\vec{v}|}} f(a,b)}$$

This is called the **directional derivative** of the function f at the point (a,b) in the direction $\vec{v}$.

Implications

We have just seen that the instantaneous rate of change of f per unit distance as we leave (a,b) moving in direction $\vec{v}$ is

$$\vec{\nabla} f(a,b)\cdot\tfrac{\vec{v}}{|\vec{v}|}=|\vec{\nabla} f(a,b)|\cos\theta$$

where θ is the angle between the gradient vector $\vec{\nabla} f(a,b)$ and the direction vector $\vec{v}$. Since $\cos\theta$ is always between -1 and $+1$

- the direction of maximum rate of increase is that having $\theta=0$. So to get maximum rate of increase per unit distance, as you leave (a,b), you should move in the same direction as the gradient $\vec{\nabla} f(a,b)$. Then the rate of increase per unit distance is $|\vec{\nabla} f(a,b)|$.
- The direction of minimum (i.e. most negative) rate of increase is that having $\theta=180°$. To get minimum rate of increase per unit distance you should move in the direction opposite $\vec{\nabla} f(a,b)$. Then the rate of increase per unit distance is $-|\vec{\nabla} f(a,b)|$.
- The directions giving zero rate of increase are those perpendicular to $\vec{\nabla} f(a,b)$. If you move in a direction perpendicular to $\vec{\nabla} f(a,b)$, $f(x,y)$ remains constant as you leave (a,b). That is, at that instant you are moving along the level curve $f(x,y)=f(a,b)$. So $\vec{\nabla} f(a,b)$ is perpendicular to the level curve $f(x,y)=f(a,b)$. We have already seen the corresponding statement in three dimensions. A good way to find a vector normal to the surface $F(x,y,z)=0$ at the point (a,b,c) is to compute the gradient $\vec{\nabla} F(a,b,c)$.

An example

Let

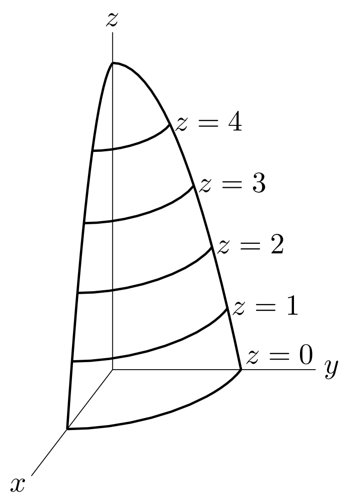
$$f(x,y)=5-x^2-2y^2 \qquad (x_0,y_0)=\big(-1,-1\big)$$

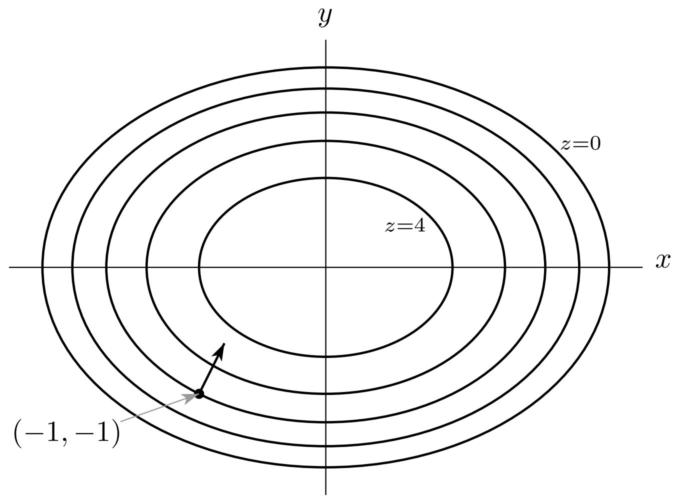
Note that for any fixed $z_0<5$, $f(x,y)=z_0$ is the ellipse $x^2+2y^2=5-z_0$. This ellipse has x–semi–axis $\sqrt{5-z_0}$ and y–semi–axis $\sqrt{\frac{5-z_0}{2}}$. The graph $z=f(x,y)$ consists of a bunch of horizontal ellipses stacked one on top of each other, starting with a point on the z axis when $z_0=5$ and increasing in size as z_0 decreases, as illustrated in the first figure below. Several level curves are sketched in the second figure below. The gradient vector

$$\boldsymbol{\nabla} f(x_0,y_0)=\big(-2x,-4y\big)\big|_{(-1,-1)}=(2,4)=2(1,2)$$

at (x_0,y_0) is also illustrated in the second sketch. We have that, at (x_0,y_0)

- the direction of maximum rate of increase is $\frac{1}{\sqrt{5}}(1,2)$ and the maximum rate of increase is $|(2,4)|=2\sqrt{5}$.
- the direction of minimum rate of increase is $-\frac{1}{\sqrt{5}}(1,2)$ and that minimum rate is $-|(2,4)|=-2\sqrt{5}$.
- the directions giving zero rate of increase are perpendicular to $\boldsymbol{\nabla} f(x_0,y_0)$, that is $\pm\frac{1}{\sqrt{5}}(2,-1)$. These are the directions of the tangent vector at (x_0,y_0) to the level curve of f through (x_0,y_0).


z
z = 4
z = 3
z = 2
z = 1
z = 0
y
x


y
z=0
z=4
x
(−1, −1)