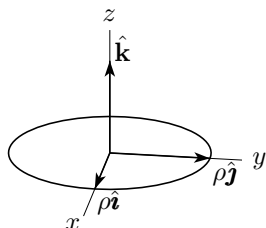


## Parametrizing Circles

These notes discuss a simple strategy for parametrizing circles in three dimensions. We start with the circle in the  $xy$ -plane that has radius  $\rho$  and is centred on the origin. This is easy to parametrize:



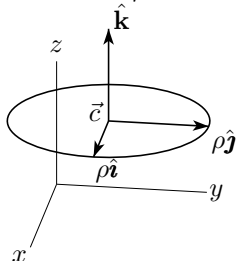
$$\vec{r}(t) = \rho \cos t \hat{i} + \rho \sin t \hat{j} \quad 0 \leq t \leq 2\pi$$

Note that we can check that  $\vec{r}(t)$  lies on the desired circle by checking, firstly, that  $\vec{r}(t)$  lies in the correct plane (in this case, the  $xy$ -plane) and, secondly, that the distance from  $\vec{r}(t)$  to the centre of the circle is  $\rho$ :

$$|\vec{r}(t) - \vec{0}| = |\rho \cos t \hat{i} + \rho \sin t \hat{j}| = \sqrt{(\rho \cos t)^2 + (\rho \sin t)^2} = \rho$$

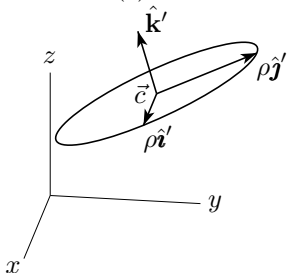
since  $\sin^2 t + \cos^2 t = 1$ .

Now let's move the circle so that its centre is at some general point  $\vec{c}$ . To parametrize this new circle, which still has radius  $\rho$  and which is still parallel to the  $xy$ -plane, we just translate by  $\vec{c}$ :



$$\vec{r}(t) = \vec{c} + \rho \cos t \hat{i} + \rho \sin t \hat{j} \quad 0 \leq t \leq 2\pi$$

Finally, let's consider a circle in general position. The secret to parametrizing a general circle is to replace  $\hat{i}$  and  $\hat{j}$  by two new vectors  $\hat{i}'$  and  $\hat{j}'$  which (a) are unit vectors, (b) are parallel to the plane of the desired circle and (c) are mutually perpendicular.



$$\vec{r}(t) = \vec{c} + \rho \cos t \hat{i}' + \rho \sin t \hat{j}' \quad 0 \leq t \leq 2\pi$$

To check that this is correct, observe that

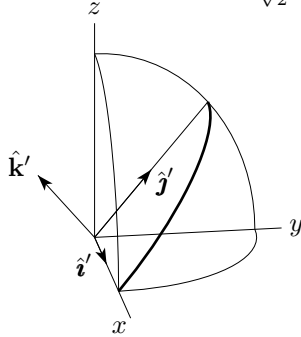
- $\vec{r}(t) - \vec{c}$  is parallel to the plane of the desired circle because  $\vec{r}(t) - \vec{c} = \rho \cos t \hat{i}' + \rho \sin t \hat{j}'$  and both  $\hat{i}'$  and  $\hat{j}'$  are parallel to the plane of the desired circle
- $\vec{r}(t) - \vec{c}$  is of length  $\rho$  for all  $t$  because

$$\begin{aligned} |\vec{r}(t) - \vec{c}|^2 &= (\vec{r}(t) - \vec{c}) \cdot (\vec{r}(t) - \vec{c}) \\ &= (\rho \cos t \hat{i}' + \rho \sin t \hat{j}') \cdot (\rho \cos t \hat{i}' + \rho \sin t \hat{j}') \\ &= \rho^2 \cos^2 t \hat{i}' \cdot \hat{i}' + \rho^2 \sin^2 t \hat{j}' \cdot \hat{j}' + 2\rho \cos t \sin t \hat{i}' \cdot \hat{j}' \\ &= \rho^2 (\cos^2 t + \sin^2 t) = \rho^2 \end{aligned}$$

since  $\hat{i}' \cdot \hat{i}' = \hat{j}' \cdot \hat{j}' = 1$  ( $\hat{i}'$  and  $\hat{j}'$  are both unit vectors) and  $\hat{i}' \cdot \hat{j}' = 0$  ( $\hat{i}'$  and  $\hat{j}'$  are perpendicular).

To find such a parametrization in practice, we need to find the centre  $\vec{c}$  of the circle, the radius  $\rho$  of the circle and two mutually perpendicular unit vectors,  $\hat{i}'$  and  $\hat{j}'$ , in the plane of the circle. It is often easy to find at least one point  $\vec{p}$  on the circle. Then we can take  $\hat{i}' = \frac{\vec{p}-\vec{c}}{|\vec{p}-\vec{c}|}$ . It is also often easy to find a unit vector,  $\hat{k}'$ , that is normal to the plane of the circle. Then we can choose  $\hat{j}' = \hat{k}' \times \hat{i}'$ .

**Example 1** Let  $C$  be the intersection of the sphere  $x^2 + y^2 + z^2 = 4$  and the plane  $z = y$ . The intersection of any plane with any sphere is a circle. The plane in question passes through the centre of the sphere, so  $C$  has the same centre and same radius as the sphere. So  $C$  has radius 2 and centre  $(0, 0, 0)$ . The point  $(2, 0, 0)$  satisfies both  $x^2 + y^2 + z^2 = 4$  and  $z = y$  and so is on  $C$ . We may choose  $\hat{i}'$  to be the unit vector in the direction from the centre  $(0, 0, 0)$  of the circle towards  $(2, 0, 0)$ . Namely  $\hat{i}' = (1, 0, 0)$ . Since the plane of the circle is  $z - y = 0$ , the vector  $\vec{\nabla}(z - y) = (0, -1, 1)$  is perpendicular to the plane of  $C$ . So we may take  $\hat{k}' = \frac{1}{\sqrt{2}}(0, -1, 1)$ . Then  $\hat{j}' = \hat{k}' \times \hat{i}' = \frac{1}{\sqrt{2}}(0, -1, 1) \times (1, 0, 0) = \frac{1}{\sqrt{2}}(0, 1, 1)$ . Subbing in  $\vec{c} = (0, 0, 0)$ ,  $\rho = 2$ ,  $\hat{i}' = (1, 0, 0)$  and  $\hat{j}' = \frac{1}{\sqrt{2}}(0, 1, 1)$  gives



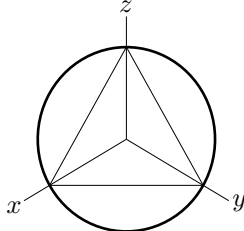
$$\vec{r}(t) = 2 \cos t (1, 0, 0) + 2 \sin t \frac{1}{\sqrt{2}}(0, 1, 1) = 2 \left( \cos t, \frac{\sin t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}} \right) \quad 0 \leq t \leq 2\pi$$

To check this, note that  $x = 2 \cos t$ ,  $y = \sqrt{2} \sin t$ ,  $z = \sqrt{2} \sin t$  satisfies both  $x^2 + y^2 + z^2 = 4$  and  $z = y$ .

**Example 2** Let  $C$  be the circle that passes through the three points  $(3, 0, 0)$ ,  $(0, 3, 0)$  and  $(0, 0, 3)$ . All three points obey  $x + y + z = 3$ . So the circle lies in the plane  $x + y + z = 3$ . We guess, by symmetry, or by looking at the figure below, that the centre of the circle is at the centre of mass of the three points, which is  $\frac{1}{3}[(3, 0, 0) + (0, 3, 0) + (0, 0, 3)] = (1, 1, 1)$ . We can check this by checking that  $(1, 1, 1)$  is equidistant from the three points:

$$\begin{aligned} |(3, 0, 0) - (1, 1, 1)| &= |(2, -1, -1)| = \sqrt{6} \\ |(0, 3, 0) - (1, 1, 1)| &= |(-1, 2, -1)| = \sqrt{6} \\ |(0, 0, 3) - (1, 1, 1)| &= |(-1, -1, 2)| = \sqrt{6} \end{aligned}$$

This tells us both that  $(1, 1, 1)$  is indeed the centre and that the radius of  $C$  is  $\sqrt{6}$ . We may choose  $\hat{i}'$  to be the unit vector in the direction from the centre  $(1, 1, 1)$  of the circle towards  $(3, 0, 0)$ . Namely  $\hat{i}' = \frac{1}{\sqrt{6}}(2, -1, -1)$ . Since the plane of the circle is  $x + y + z = 3$ , the vector  $\vec{\nabla}(x + y + z) = (1, 1, 1)$  is perpendicular to the plane of  $C$ . So we may take  $\hat{k}' = \frac{1}{\sqrt{3}}(1, 1, 1)$ . Then  $\hat{j}' = \hat{k}' \times \hat{i}' = \frac{1}{\sqrt{18}}(1, 1, 1) \times (2, -1, -1) = \frac{1}{\sqrt{18}}(0, 3, -3) = \frac{1}{\sqrt{2}}(0, 1, -1)$ . Subbing in  $\vec{c} = (1, 1, 1)$ ,  $\rho = \sqrt{6}$ ,  $\hat{i}' = \frac{1}{\sqrt{6}}(2, -1, -1)$  and  $\hat{j}' = \frac{1}{\sqrt{2}}(0, 1, -1)$  gives



$$\begin{aligned} \vec{r}(t) &= (1, 1, 1) + \sqrt{6} \cos t \frac{1}{\sqrt{6}}(2, -1, -1) + \sqrt{6} \sin t \frac{1}{\sqrt{2}}(0, 1, -1) \\ &= (1 + 2 \cos t, 1 - \cos t + \sqrt{3} \sin t, 1 - \cos t - \sqrt{3} \sin t) \quad 0 \leq t \leq 2\pi \end{aligned}$$

To check this, note that  $\vec{r}(0) = (3, 0, 0)$ ,  $\vec{r}(\frac{2\pi}{3}) = (0, 3, 0)$  and  $\vec{r}(\frac{4\pi}{3}) = (0, 0, 3)$  since  $\cos \frac{2\pi}{3} = \cos \frac{4\pi}{3} = -\frac{1}{2}$ ,  $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$  and  $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ .