# Warnings

These notes highlight number of common, but serious, elementary differential equations errors.

**Warning 1.** You can always check whether or not a given function is a solution to a given differential equation by substituting the function into the differential equation.

**Warning 2.** The formula

$$\mu(t) = \exp\left[\textstyle\int p(t)\,dt\right]$$

for an integrating factor for the differential equation $y'(t)+p(t)y(t) = g(t)$ is valid only when the coefficient of $y'(t)$ in the equation is 1.

**Discussion.** For example, the differential equation

$$ty'(t) + 2y(t) = t^3$$

does **not** have integrating factor $\mu(t) = \exp\left[\int 2\,dt\right] = e^{2t}$ and does **not** have general solution

$$y(t) = \tfrac{1}{\mu(t)}\left[\textstyle\int g(t)\mu(t)\,dt \ + \ C\right] = e^{-2t}\left[\int t^3e^{2t}\,dt \ + \ C\right]$$

A good way to catch this error is to

a) Multiply the equation by the function $\mu(t)$ that you think is the integrating factor. In this example

$$te^{2t}y'(t) + 2e^{2t}y(t) = t^3e^{2t}$$

b) Then check that the left hand side really is $\frac{d}{dt}\left[\mu(t)y(t)\right]$. In this example

$$\tfrac{d}{dt}\left[\mu(t)y(t)\right] = \tfrac{d}{dt}\left[e^{2t}y(t)\right] = e^{2t}y'(t) + 2e^{2t}y(t)$$

is not the same as the left hand side $te^{2t}y'(t) + 2e^{2t}y(t)$.

The safe way to solve this equation is to first divide through by $t$ to make the coefficient of $y'(t)$ in the equation 1. Then

$$y'(t) + \tfrac{2}{t}y(t) = t^2$$

so that

$$\mu(t) = \exp\left[\int \tfrac{2}{t}\,dt\right] = e^{2\ln t} = t^2$$

Multiplying $y'(t) + \frac{2}{t}y(t) = t^2$ by $t^2$ gives $t^2y'(t) + 2ty(t) = t^4$, whose left hand side really is

$$\tfrac{d}{dt}\big[\mu(t)y(t)\big] = \tfrac{d}{dt}\big[t^2y(t)\big] = t^2y'(t) + 2ty(t)$$

so the equation really is $\frac{d}{dt}\big[t^2y(t)\big] = t^4$ and $t^2y(t) = \int t^4\,dt = \frac{t^5}{5} + C$.

**Warning 3.** Note that the arbitrary constant $C$ in the canned formula

$$y(t) = \tfrac{1}{\mu(t)}\Big[\int g(t)\mu(t)\,dt \ + \ C\Big] \qquad \text{with} \qquad \mu(t) = \exp\left[\int p(t)\,dt\right]$$

for the solution of the differential equation $y'(t) + p(t)y(t) = g(t)$ is **inside** the brackets.

**Warning 4.** $\ln(a+b) \neq \ln a + \ln b$

**Discussion.** For example, suppose you solve $\frac{1}{x}e^y\frac{dy}{dx} - 2 = 0$, $y(0) = 2$ as a separable equation.

$$\begin{aligned}
\tfrac{1}{x}e^y\tfrac{dy}{dx} - 2 = 0 &\iff e^y\tfrac{dy}{dx} = 2x \\
&\iff \int e^y\,dy = \int 2x\,dx \\
&\iff e^y = x^2 + C \qquad \text{(The arbitrary constant } C \text{ enters } \textbf{here}.\text{)} \\
y(0) = 2 \ \Rightarrow\ C = e^2 \ &\Rightarrow\ e^y = x^2 + e^2
\end{aligned}$$

Taking the ln of both sides gives $y = \ln\left(x^2 + e^2\right)$ **not** $y = 2\ln x + 2$.

**Warning 5.** When generating an approximate solution to an initial value problem using Euler's method or the improved Euler method or the Runge-Kutta method, **be careful about how many steps you use**.

**Discussion.** For example, suppose that you wish to find an approximate value for $y(1.3)$ when $y(t)$ obeys $y'(t) = \ln y(t)$, $y(1) = 2$. If you use step size $h = 0.15$, $y_0 = y(1)$, $y_1$ is an approximate value for $y(1.15)$ and $y_2$ is an approximate value for $y(1.30)$. So it is appropriate to do two (not three) steps.

**Warning 6.** When generating an approximate solution to an initial value problem using Euler's method or the improved Euler method or the Runge-Kutta method, be careful only to use step sizes $h$ which give an **integer number of steps**.

**Discussion.** For example, suppose that you wish to find an approximate value for $y(1.3)$ when $y(t)$ obeys $y'(t) = \ln y(t)$, $y(1) = 2$. If you use step size $h = 0.07$, $y_0 = y(1)$, $y_1$ is an approximate value for $y(1.07)$, $y_2$ is an approximate value for $y(1.14)$, $y_3$ is an approximate value for $y(1.21)$, $y_4$ is an approximate value for $y(1.28)$ and $y_5$ is an approximate value for $y(1.35)$. **No** $y_n$ is an approximate value for $y(1.3)$. In this example the step size must be of the form $h = \frac{1.3-1.0}{n}$ for some integer $n$.

**Warning 7.** The canned Richardson extrapolation formulae

$$\begin{aligned} A &= \frac{2^k A(h/2) - A(h)}{2^k - 1} + O(h^{k+1}) \\ B(h) &= \frac{2^k A(h/2) - A(h)}{2^k - 1} \end{aligned}$$

may only be used if you have approximate data for two step sizes (namely $h$ and $h/2$) with the larger of the two being **exactly twice** the smaller of the two.

**Discussion.** Otherwise, you have to derive your own formulae by solving

$$\begin{aligned} A &= A(h) + Kh^k + O(h^{k+1}) \\ A &= A(h') + Kh'^k + O(h'^{k+1}) \end{aligned}$$

simultaneously for $A$ and $K$.

**Warning 8.** Consider the problem of finding the solution to

$$y'' - 3y' + 2y = 8, \quad y(0) = 4, \; y'(0) = 1$$

The general solution of $y'' - 3y' + 2y = 8$ is the sum of two pieces, namely $c_1e^{2t} + c_2e^t$, which is the general solution of $y'' - 3y + 2y = 0$, and 4, which is a particular solution of

$y'' - 3y' + 2y = 8$. The initial conditions $y(0) = 4,\ y'(0) = 1$ apply to the full $y$ not just the $c_1e^{2t} + c_2e^t$ part of it. That is, the equations which determine $c_1$ and $c_2$ are

$$\begin{aligned} y(0) &= \big[c_1e^{2t} + c_2e^t + 4\big]_{t=0} &&= c_1 + c_2 + 4 = 4 \\ y'(0) &= \big[2c_1e^{2t} + c_2e^t\big]_{t=0} &&= 2c_1 + c_2 \qquad = 1 \end{aligned}$$

and **not**

$$\begin{aligned} y(0) &= \big[c_1e^{2t} + c_2e^t\big]_{t=0} &&= c_1 + c_2 \quad = 4 \\ y'(0) &= \big[2c_1e^{2t} + c_2e^t\big]_{t=0} &&= 2c_1 + c_2 = 1 \end{aligned}$$

**Warning 9.** By definition the vector all of whose entries are zero is **never** an eigenvector. Most of the time, when you get zero while attempting to find an eigenvector, you have made an error computing the eigenvalues.