Richardson Extrapolation

There are many approximation procedures in which one first picks a step size $h$ and then generates an approximation $A(h)$ to some desired quantity $A$. Often the order of the error generated by the procedure is known. In other words

$$A = A(h) + K h^k + K' h^{k+1} + K'' h^{k+2} + \cdots$$

with $k$ being some known constant and $K$, $K'$, $K''$, $\cdots$ being some other (usually unknown) constants. For example, $A$ might be the value $y(t_f)$ at some final time $t_f$ for the solution to an initial value problem $y' = f(t, y)$, $y(t_0) = y_0$. Then $A(h)$ might be the approximation to $y(t_f)$ produced by Euler’s method with step size $h$. In this case $k = 1$. If the improved Euler’s method is used $k = 2$. If Runge-Kutta is used $k = 4$.

The notation $O(h^{k+1})$ is conventionally used to stand for “a sum of terms of order $h^{k+1}$ and higher”. So the above equation may be written

$$A = A(h) + K h^k + O(h^{k+1}) \quad (1)$$

If we were to drop the, hopefully tiny, term $O(h^{k+1})$ from this equation, we would have one linear equation, $A = A(h) + K h^k$, in the two unknowns $A, K$. But this is really a different equation for each different value of $h$. We can get a second such equation just by using a different step size. Then the two equations may be solved, yielding approximate values of $A$ and $K$. This approximate value of $A$ constitutes a new improved approximation, $B(h)$, for the exact $A$. We do this now. Taking $2^k$ times

$$A = A(h/2) + K (h/2)^k + O(h^{k+1}) \quad (2)$$

(note that, in equations (1) and (2), the symbol “$O(h^{k+1})$” is used to stand for two different sums of terms of order $h^{k+1}$ and higher) and subtracting equation (1) gives

$$(2^k - 1) A = 2^k A(h/2) - A(h) + O(h^{k+1})$$

$$A = \frac{2^k A(h/2) - A(h)}{2^k - 1} + O(h^{k+1})$$
Hence if we define

$$B(h) = \frac{2^k A(h/2) - A(h)}{2^k - 1}$$

(3)

then

$$A = B(h) + O(h^{k+1})$$

(4)

and we have generated an approximation whose error is of order $k+1$, one better than $A(h)$’s.

One widely used numerical integration algorithm, called Romberg integration, applies this formula repeatedly to the trapezoidal rule.

**Example**

$$A = y(1) = 64.897803$$

where $y(t)$ obeys $y(0) = 1$, $y' = 1 - t + 4y$.

$A(h) =$ approximate value for $y(1)$ given by improved Euler with step size $h$.

$B(h) = \frac{2^k A(h/2) - A(h)}{2^k - 1}$ with $k = 2$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$A(h)$</th>
<th>%</th>
<th>#</th>
<th>$B(h)$</th>
<th>%</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td>.1</td>
<td>59.938</td>
<td>7.6</td>
<td>20</td>
<td>64.587</td>
<td>48</td>
<td>60</td>
</tr>
<tr>
<td>.05</td>
<td>63.424</td>
<td>2.3</td>
<td>40</td>
<td>64.856</td>
<td>65</td>
<td>120</td>
</tr>
<tr>
<td>.025</td>
<td>64.498</td>
<td>.62</td>
<td>80</td>
<td>64.892</td>
<td>.0083</td>
<td>240</td>
</tr>
<tr>
<td>.0125</td>
<td>64.794</td>
<td>.04</td>
<td>160</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The “%” column gives the percentage error and the “#” column gives the number of evaluations of $f(t, y)$ used.

Similarly, by subtracting equation (2) from equation (1), we can find $K$.

$$0 = A(h) - A(h/2) + K h^k \left(1 - \frac{1}{2^k}\right) + O(h^{k+1})$$

$$K = \frac{A(h/2) - A(h)}{h^k \left(1 - \frac{1}{2^k}\right)} + O(h)$$

Once we know $K$ we can estimate the error in $A(h/2)$ by

$$E(h/2) = A - A(h/2)$$

$$= K(h/2)^k + O(h^{k+1})$$

$$= \frac{A(h/2) - A(h)}{2^k - 1} + O(h^{k+1})$$

If this error is unacceptably large, we can use

$$E(h) \cong Kh^k$$

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to determine a step size $h$ that will give an acceptable error. This is the basis for a number of algorithms that incorporate automatic step size control.

Note that $\frac{A(h/2) - A(h)}{2} = B(h) - A(h/2)$. One cannot get a still better guess for $A$ by combining $B(h)$ and $E(h/2)$.

**Example.** Suppose that we wished to use improved Euler to find a numerical approximation to $A = y(1)$, where $y$ is the solution to the initial value problem

$$y' = y - 2t \quad y(0) = 3$$

Suppose further that we are aiming for an error of $10^{-6}$. If we run improved Euler with step size 0.2 (5 steps) and again with step size 0.1 (10 steps) we get the approximate values $A(0.2) = 6.70270816$ and $A(0.1) = 6.71408085$. Since improved Euler has $k = 2$, These approximate values obey

$$A = A(0.2) + K(0.2)^2 + \text{higher order} = 6.70270816 + K(0.2)^2 + \text{higher order}$$

$$A = A(0.1) + K(0.1)^2 + \text{higher order} = 6.71408085 + K(0.1)^2 + \text{higher order}$$

Subtracting

$$0 = 6.70270816 + K(0.2)^2 - 6.71408085 - K(0.1)^2 + \text{higher order} \approx -0.01137269 + 0.03K$$

so that

$$K \approx \frac{0.01137269}{0.03} = 0.38$$

The error for step size $h$ is $Kh^2 + O(h^3)$, so to achieve an error of $10^{-6}$ we need

$$Kh^2 + O(h^3) = 10^{-6} \quad \Rightarrow \quad 0.38 h^2 \approx 10^{-6} \quad \Rightarrow \quad h \approx \sqrt{\frac{10^{-6}}{0.38}} = 0.001622 = \frac{1}{616.5}$$

If we run improved Euler with step size $\frac{1}{617}$ we get the approximate value $A(\frac{1}{617}) = 6.71828064$. In this illustrative, and purely artifical, example, we can solve the intial value problem exactly. The solution is $y(t) = 2+2t+e^t$, so that the exact value of $y(1) = 6.71828183$, to eight decimal places, and the error in $A(\frac{1}{617})$ is 0.00000119.