December 1999 MATH 256 Exam Solutions

1) (a) Suppose that the area $A(t)$ of a disk is changing at a rate proportional to its radius $R(t)$. If $A(0) = 2$ and $A(1) = 1$, when will the disk disappear?
(b) Suppose that the rate of change of $A(t)$ at time t is equal to $\pi\sqrt{R(t)}$. If $A(0) = \pi$, at what time will $A(t) = 2$?

Solution. (a) Since $R(t)$ is proportional to $\sqrt{A(t)}$,

$$\tfrac{dA}{dt} = \kappa\sqrt{A(t)} \quad\Longrightarrow\quad \frac{dA}{\sqrt{A(t)}} = \kappa\, dt \quad\Longrightarrow\quad 2\sqrt{A} = \kappa t + C$$

Imposing the conditions at time 0 and time 1

$$\begin{aligned}
A(0) = 2 &\;\Rightarrow\; 2\sqrt{2} = C \\
A(1) = 1 &\;\Rightarrow\; 2 = \kappa + C \;\Rightarrow\; \kappa = 2 - 2\sqrt{2} \\
A(t) = 0 &\;\Rightarrow\; 0 = \kappa t + C \;\Rightarrow\; t = -\tfrac{C}{\kappa} = -\tfrac{2\sqrt{2}}{2-2\sqrt{2}} = \boxed{\tfrac{\sqrt{2}}{\sqrt{2}-1} = 3.4142}
\end{aligned}$$

(b) $\boxed{\text{Never}}$. The area, $A(t)$ increases with time, since $\frac{dA}{dt} = \pi\sqrt{R(t)} = \pi^{3/4}A(t)^{1/4} > 0$, and $A(t)$ starts at $\pi > 2$ when $t = 0$.

2) Suppose that the numerical calculation of the solution to an initial value problem $y' = f(y,t)$, $y(0) = 0$, using one of Euler, Improved Euler of Runge–Kutta methods yield the following results:

Step size h	Approximation for $y(1)$
0.50	1.6325
0.10	1.4947
0.05	1.4904

a) Which method was used? How do you know?
b) Given this data, what is your best estimate for $y(1)$?

Solution. a) Let A be the exact value of $y(1)$ and $A(h)$ be the approximation for step size h. Then

$$A = A(h) + Kh^k + O(h^{k+1})$$

so that

$$\begin{aligned}
A &= 1.6325 + K(0.5)^k + O(h^{k+1}) && (1)\\
A &= 1.4947 + K(0.1)^k + O(h^{k+1}) && (2)\\
A &= 1.4904 + K(0.05)^k + O(h^{k+1}) && (3)\\
0 &= 0.1378 + K\big[(0.5)^k - (0.1)^k\big] + O(h^{k+1}) && (1)-(2)\\
0 &= 0.0043 + K\big[(0.1)^k - (0.05)^k\big] + O(h^{k+1}) && (2)-(3)
\end{aligned}$$

and

$$-0.1378 \approx K\big[(0.5)^k - (0.1)^k\big] \tag{4}$$
$$-0.0043 \approx K\big[(0.1)^k - (0.05)^k\big] \tag{5}$$

Dividing the two equations

$$32.0465 = \frac{(0.5)^k - (0.1)^k}{(0.1)^k - (0.05)^k} = \begin{cases} 8 & \text{if } k = 1 \\ 32 & \text{if } k = 2 \\ 665.6 & \text{if } k = 4 \end{cases}$$

So $\boxed{k = 2}$.
b)

$$\begin{aligned} 3A &= 4 \times 1.4904 - 1.4947 + \text{higher order} && 4(3) - (2) \\ A &\approx \tfrac{4\times1.4904-1.4947}{3} = \boxed{1.4890} \end{aligned}$$

3) Find the general solution of

$$y'' + 9y = x\cos(3x)$$

Solution. The general solution of $y''+9y = 0$ is $c_1\cos(3x)+c_2\sin(3x)$ or $d_1e^{i3x}+d_2e^{-i3x}$. We still have to find a particular solution.
Method 1 Note that $y(x) = Ax\cos(3x) + Bx\sin(3x) + C\cos(3x) + D\sin(3x)$ will not work, because $C\cos(3x)$ and $D\sin(3x)$ satisfy $y'' + 9y = 0$. So try

$$\begin{aligned} y(x) &= \quad Ax^2\cos(3x) + \quad Bx^2\sin(3x) + Cx\cos(3x) + Dx\sin(3x) \\ y'(x) &= \quad 3Bx^2\cos(3x) - 3Ax^2\sin(3x) + (2A+3D)x\cos(3x) + (2B-3C)x\sin(3x) \\ &\qquad + C\cos(3x) + D\sin(3x) \\ y''(x) &= -9Ax^2\cos(3x) - 9Bx^2\sin(3x) + (12B-9C)x\cos(3x) + (-12A-9D)x\sin(3x) \\ &\qquad + (2A+6D)\cos(3x) + (2B-6C)\sin(3x) \end{aligned}$$

Hence

$$\begin{aligned} y'' + 9y &= 12Bx\cos(3x) - 12Ax\sin(3x) + (2A+6D)\cos(3x) + (2B-6C)\sin(3x) \\ &= x\cos(3x) \iff 12B = 1,\ 12A = 0,\ 2A+6D = 0,\ 2B-6C = 0 \end{aligned}$$

so that $A = D = 0,\ B = \frac{1}{12},\ C = \frac{1}{36}$ and

$$\boxed{y = c_1\cos(3x) + c_2\sin(3x) + \tfrac{1}{12}x^2\sin(3x) + \tfrac{1}{36}x\cos(3x)}$$

Method 2 First find a particular solution to $y'' + 9y = xe^{i3x}$ by trying $y = Ax^2e^{i3x} + Bxe^{i3x}$. (Note that $y = Axe^{i3x} + Be^{i3x}$ will not workd since Be^{i3x} satisfies $y'' + 9y = 0$.)

$$\begin{aligned}
y(x) &= \quad Ax^2e^{i3x} + Bxe^{i3x} \\
y'(x) &= 3iAx^2e^{i3x} + (2A + 3iB)xe^{i3x} + Be^{i3x} \\
y''(x) &= -9Ax^2e^{i3x} + (12iA - 9B)xe^{i3x} + (2A + 6iB)e^{i3x} \\
y''(x) + 9y(x) &= 12iAxe^{i3x} + (2A + 6iB)e^{i3x} \\
&= xe^{i3x} \iff 12iA = 1,\ 2A + 6iB = 0 \iff A = \tfrac{1}{12i},\ B = \tfrac{1}{36}
\end{aligned}$$

So

$$\begin{array}{lll}
y = \frac{1}{12i}x^2e^{i3x} + \frac{1}{36}xe^{i3x} & \text{obeys } y'' + 9y = xe^{i3x} & (1) \\
y = -\frac{1}{12i}x^2e^{-i3x} + \frac{1}{36}xe^{-i3x} & \text{obeys } y'' + 9y = xe^{-i3x} & i \to -i(2) \\
y = \frac{1}{12}x^2\sin(3x) + \frac{1}{36}x\cos(3x) & \text{obeys } y'' + 9y = x\cos(3x) & \frac{1}{2}(1) + \frac{1}{2}(2)
\end{array}$$

This is the same particular solution as given by method 1.

4) Solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with initial condition $\vec{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Solution. Set $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Then

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) \ \Rightarrow\ \boxed{\lambda = 3, -1}$$

$$\text{For eigenvalue } \lambda = 3,\ A - \lambda I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \implies \boxed{\text{eigenvectors } c\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, c \neq 0$$

$$\text{For eigenvalue } \lambda = -1,\ A - \lambda I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \implies \boxed{\text{eigenvectors } c\begin{bmatrix} 1 \\ -1 \end{bmatrix}}, c \neq 0$$

For a particular solution, try $\vec{x} = \vec{a}$, a constant. It solves the differential equation if

$$\vec{0} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{a} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies a_1 = \tfrac{1}{3},\ a_2 = -\tfrac{2}{3}$$

So the general solution to $\vec{x}'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \tfrac{1}{3} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The initial condition is satisfied if

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \tfrac{1}{3} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \iff \begin{array}{r} c_1 + c_2 = \frac{5}{3} \\ c_1 - c_2 = \frac{2}{3} \end{array} \iff \begin{array}{r} c_1 = \frac{7}{6} \\ c_2 = \frac{1}{2} \end{array}$$

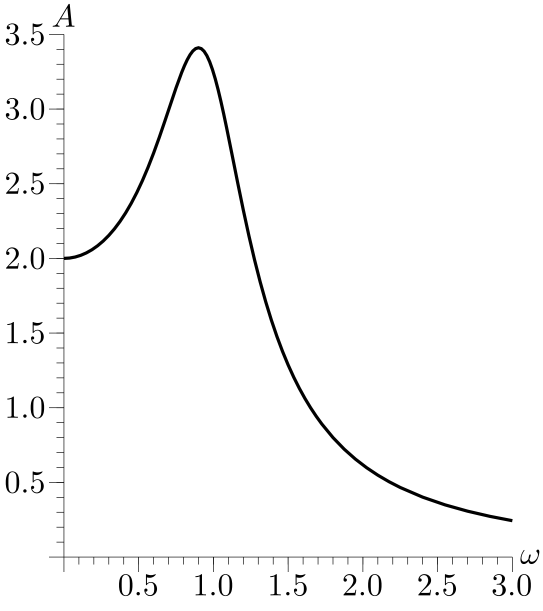
The solution is

$$\boxed{\vec{x}(t) = \tfrac{7}{6} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \tfrac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \tfrac{1}{3} \begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

5) Suppose that u(t) is the displacement from equilibrium of a mass on a spring, subject to an external force $2\cos(\omega t)$. Thus u satisfies the equation

$$mu'' + \gamma u' + ku = 2\cos(\omega t)$$

a) If $k - m\omega^2 + i\omega\gamma = 2e^{i\pi/4}$, what is $u(t)$ after a long time has passed?
b) Suppose that $k = m = 1$ and the amplitude $A(\omega)$ of the steady state solution has the graph below. What is the approximate value of γ?
c) Suppose that $k = m = 1$ and the maximum amplitude occurs at $\omega = 0$. What can you say about γ?



Solution. a) We are to find a particular solution to $mu'' + \gamma u' + ku = 2\cos(\omega t) = e^{i\omega t} + e^{-i\omega t}$. To do so, we first find a particular solution for $mu'' + \gamma u' + ku = e^{i\omega t}$, of the form $u(t) = Be^{i\omega t}$.

$$mu'' + \gamma u' + ku = B\big(- m\omega^2 + i\gamma\omega + k\big)e^{i\omega t} = e^{i\omega t} \quad \Longrightarrow \quad B = \tfrac{1}{k - m\omega^2 + i\gamma\omega}$$

But any complex number $x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$. Hence

$$k - m\omega^2 + i\gamma\omega = Re^{i\delta} \qquad \text{where } R = \sqrt{(k - m\omega^2)^2 + (\omega\gamma)^2},\ \delta = \tan^{-1}\tfrac{\gamma\omega}{k - m\omega^2}$$

and

$$\begin{aligned} u &= Be^{i\omega t} = \tfrac{1}{Re^{i\delta}}e^{i\omega t} = \tfrac{1}{R}e^{i(\omega t - \delta)} && \text{obeys } mu'' + \gamma u' + ku = e^{i\omega t} && (1)\\ u &= \tfrac{1}{R}e^{-i(\omega t - \delta)} && \text{obeys } mu'' + \gamma u' + ku = e^{-i\omega t} && i \to -i(2)\\ u &= \tfrac{2}{R}\cos(\omega t - \delta) && \text{obeys } mu'' + \gamma u' + ku = 2\cos(\omega t) && (1) + (2) \end{aligned}$$

In the case $k - m\omega^2 + i\omega\gamma = 2e^{i\pi/4}$, we have $R = 2$, $\delta = \frac{\pi}{4}$ so that

$$\boxed{u(t) = \cos\left(\omega t - \tfrac{\pi}{4}\right)}$$

b) When $k = m = 1$, the amplitude of the steady state solution is

$$A(\omega) = \tfrac{2}{R} = \tfrac{2}{\sqrt{(k - m\omega^2)^2 + (\omega\gamma)^2}} = \tfrac{2}{\sqrt{(1 - \omega^2)^2 + (\omega\gamma)^2}}$$

This amplitude is a maximum when the denominator is a minimum. That is, when

$$0 = \tfrac{d}{d\omega}\left[(1 - \omega^2)^2 + (\omega\gamma)^2\right] = -4\omega(1 - \omega^2) + 2\omega\gamma^2 \quad \Longrightarrow \quad \omega = 0 \text{ or } \gamma^2 = 2(1 - \omega^2)$$

In the graph, the maximum amplitude is at about $\omega = 0.9$, so

$$\gamma \approx \boxed{\sqrt{2(1 - 0.9^2)} = 0.6}$$

c) If $\boxed{\gamma \geq \sqrt{2}}$,

$$\tfrac{d}{d\omega}\left[(1 - \omega^2)^2 + (\omega\gamma)^2\right] = 2\omega\left[\gamma^2 + 2\omega^2 - 2\right] \geq 0 \text{ for all } \omega$$

That is, the denominator increases with ω, so the amplitude decreases with ω. Conversely, if $\gamma < \sqrt{2}$, the denominator decreases with ω for small ω and the amplitude increases with ω for small ω.

6) Let $f(x) = x$ for $0 \leq x \leq 2$. Expand f in a cosine series. Sketch the graph of the function to which the cosine series converges.

Solution. We are given a function that is defined for $0 \leq x \leq \ell$ with $\ell = 2$, so

$$f(x) = \tfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\tfrac{k\pi x}{2}\right)$$

with, for $k \neq 0$

$$a_k = \int_0^2 x\cos\left(\tfrac{k\pi x}{2}\right)\,dx = \left[\tfrac{2}{k\pi}x\sin\left(\tfrac{k\pi x}{2}\right) + \left(\tfrac{2}{k\pi}\right)^2\cos\left(\tfrac{k\pi x}{2}\right)\right]_0^2 = \left(\tfrac{2}{k\pi}\right)^2\left[\cos(k\pi) - 1\right]$$

$$= -\tfrac{8}{k^2\pi^2}\begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even}\end{cases}$$

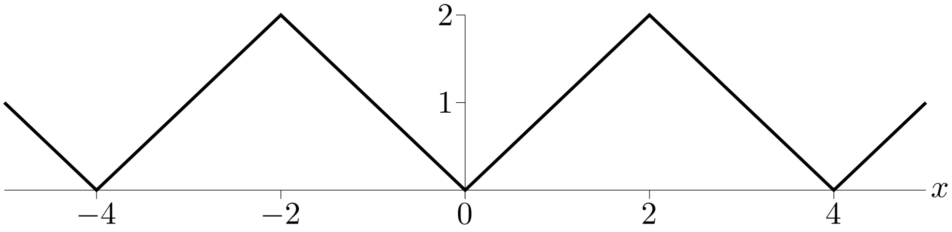
and

$$a_0 = \int_0^2 x\,dx = \tfrac{x^2}{2}\Big|_0^2 = 2$$

Hence

$$\boxed{f(x) = 1 - \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \tfrac{8}{k^2\pi^2}\cos\left(\tfrac{k\pi x}{2}\right)}$$

The right hand side is even, periodic of period 4 and takes the value x for $0 < x < 2$. So it has graph



7) Solve the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ for $0 \le x \le 1$ with the following initial and boundary conditions:

$$u(x,0) = 1 \qquad u_x(0,t) = 1 \qquad u_x(1,t) = 2$$

(*Hint:* To satisfy the boundary conditions, look for a particular solution of the differential equation having the form $u(x,t) = t + v(x)$.)

Solution. First look for a solution of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad u_x(0,t) = 1 \qquad u_x(1,t) = 2$$

of the form $u(x,t) = t + v(x)$. Subbing in

$$1 = v''(x) \qquad v'(0) = 1 \qquad v'(1) = 2 \quad\Longrightarrow\quad v'(x) = 1 + x$$

so one acceptable $v(x)$ (which is all we need) is $v(x) = x + \frac{1}{2}x^2$. Now define $w(x,t)$ by $u(x,t) = t + x + \frac{1}{2}x^2 + w(x,t)$, where $u(x,t)$ satisfies all of the conditions specified in the question. Then

$$\begin{array}{lclcl}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & \Longrightarrow & 1 + \frac{\partial w}{\partial t} = 1 + \frac{\partial^2 w}{\partial x^2} & \Longrightarrow & \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} \\
u_x(0,t) = 1 & \Longrightarrow & 1 + w_x(0,t) = 1 & \Longrightarrow & w_x(0,t) = 0 \\
u_x(1,t) = 2 & \Longrightarrow & 2 + w_x(1,t) = 2 & \Longrightarrow & w_x(1,t) = 0 \\
u(x,0) = 1 & \Longrightarrow & x + \frac{1}{2}x^2 + w(x,0) = 1 & \Longrightarrow & w(x,0) = 1 - x - \frac{1}{2}x^2
\end{array}$$

Finally, we find w by standard separation of variables. Try $w(x,t) = X(x)T(t)$.

$$\tfrac{\partial w}{\partial t} = \tfrac{\partial^2 w}{\partial x^2} \quad\Longrightarrow\quad XT' = X''T \quad\Longrightarrow\quad \frac{X''}{X} = \frac{T'}{T} = -\lambda^2, \ \ \text{const}$$

If $\lambda = 0$

$$X'' + \lambda^2 X = 0,\ X'(0) = 0,\ X'(1) = 0 \quad\Longrightarrow\quad X = c_1 + c_2 x,\ c_2 = 0 \quad\Longrightarrow\quad X = c_1$$

and

$$T' + \lambda^2 T = 0 \quad\Longrightarrow\quad T = d_1$$

If $\lambda \neq 0$ is to give a nontrivial solution

$$\begin{aligned} & X'' + \lambda^2 X = 0,\ X'(0) = 0,\ X'(1) = 0 \\ & \quad\Longrightarrow\quad X = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x},\ i\lambda(c_1 - c_2) = 0,\ i\lambda\big(c_1 e^{i\lambda} - c_2 e^{-i\lambda}\big) = 0, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad c_1, c_2 \text{ not both zero} \\ & \quad\Longrightarrow\quad e^{i\lambda} - e^{-i\lambda} = 0 \quad\Longrightarrow\quad e^{i2\lambda} = 1 \quad\Longrightarrow\quad 2\lambda = 2k\pi \quad\Longrightarrow\quad \lambda = k\pi \end{aligned}$$

and

$$T' + \lambda^2 T = 0 \quad\Longrightarrow\quad T = d_1 e^{-\lambda^2 t} = d_1 e^{-k^2\pi^2 t}$$

So $w(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-k^2\pi^2 t}\cos(k\pi x)$ satisfies the pde and both boundary conditions for all choices of the a_k's. To satisfy the initial condition we need

$$\begin{aligned} & \tfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x) = 1 - x - \tfrac{1}{2}x^2 \\ \Longrightarrow \quad & a_k = 2\int_0^1 \big(1 - x - \tfrac{1}{2}x^2\big)\cos(k\pi x)\,dx \\ & \quad = \tfrac{2}{k\pi}\Big[\sin(k\pi x) - x\sin(k\pi x) - \tfrac{1}{k\pi}\cos(k\pi x) - \tfrac{1}{2}x^2\sin(k\pi x) \\ & \qquad\qquad - \tfrac{1}{k\pi}x\cos(k\pi x) + \tfrac{1}{k^2\pi^2}\sin(k\pi x)\Big]_0^1 \qquad \text{for } k \neq 0 \\ & \quad = \tfrac{2}{k^2\pi^2}\big[1 - 2\cos(k\pi)\big] = \tfrac{2}{k^2\pi^2}\big[1 - 2(-1)^k\big] \\ & a_0 = 2\int_0^1 \big(1 - x - \tfrac{1}{2}x^2\big)\,dx = 2\big[x - \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3\big]_0^1 = \tfrac{2}{3} \end{aligned}$$

All together

$$\boxed{u(x,t) = t + x + \tfrac{1}{2}x^2 + \tfrac{1}{3} + \textstyle\sum_{k=1}^{\infty} \tfrac{2}{k^2\pi^2}\big[1 - 2(-1)^k\big]e^{-k^2\pi^2 t}\cos(k\pi x)}$$