Periodic Extensions

If a function $f(x)$ is only defined for $0 < x < \ell$ we can get many Fourier expansions for $f$ by using the following

**Main Idea** If $F(x)$ is periodic of period $2\ell$ (and hence has a Fourier series expansion) and if $f(x) = F(x)$ for $0 < x < \ell$ then, for all $x$ between $0$ and $\ell$,

$$f(x) = F(x)$$

$\quad = \text{Fourier series for } F(x)$

Motivated by this observation we define $F(x)$ to be a **periodic extension** of $f(x)$ if

1) $F(x) = f(x)$ for $0 < x < \ell$

2) $F(x)$ is periodic of period $2\ell$

There are many periodic extensions of $f(x)$. Most of them are pretty useless. For example define $g(x) = 1$ for $0 < x < \pi$. We are not defining $g(x)$ for $x \geq \pi$ or $x \leq 0$. Then

$$G(x) = \begin{cases} 1 & \text{if } 2n\pi < x < (2n + 1)\pi \text{ for some integer } n \\ -x + (2n + 2)\pi & \text{if } (2n + 1)\pi < x < (2n + 2)\pi \text{ for some integer } n \end{cases}$$

is a periodic extension of $g$. Its graph is drawn in the figure at the end of this handout.

There are two periodic extensions for $f(x)$ that are very useful. Given $f(x)$ defined for $0 < x < \ell$ we define its **even periodic extension** $F^e(x)$ by

1) $F^e(x) = f(x)$ for $0 < x < \ell$

2) $F^e(x)$ is even (This fixes $F^e(x)$ for $-\ell < x < 0$.)

3) $F^e(x)$ has period $2\ell$ (This fixes $F^e(x)$ for all remaining $x$’s, except $x = n\pi$. )

and we define its **odd periodic extension** $F^o(x)$ by

1) $F^o(x) = f(x)$ for $0 < x < \ell$

2) $F^o(x)$ is odd

3) $F^o(x)$ has period $2\ell$

By the Main Idea we have, for all $0 < x < \ell$

$$f(x) = F^e(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \left( \frac{k\pi x}{\ell} \right)$$

$$= F^o(x) = \sum_{k=1}^{\infty} b_k \sin \left( \frac{k\pi x}{\ell} \right)$$
where
\[ a_k = \frac{2}{\ell} \int_{0}^{\ell} F^e(x) \cos \left( \frac{k\pi x}{\ell} \right) \, dx = \frac{2}{\ell} \int_{0}^{\ell} f(x) \cos \left( \frac{k\pi x}{\ell} \right) \, dx \]
\[ b_k = \frac{2}{\ell} \int_{0}^{\ell} F^o(x) \sin \left( \frac{k\pi x}{\ell} \right) \, dx = \frac{2}{\ell} \int_{0}^{\ell} f(x) \sin \left( \frac{k\pi x}{\ell} \right) \, dx \]

For example consider, again, the function \( g(x) \) which is only defined for \( 0 < x < \pi \) and takes the value \( g(x) = 1 \) for all \( 0 < x < \pi \). The even and odd periodic extensions, \( G^e(x) \) and \( G^o(x) \) of this function are graphed on the next page. Both \( G^e(x) \) and \( G^o(x) \) take the value 1 for all \( 0 < x < \pi \). Both \( G^e(x) \) and \( G^o(x) \) have period \( 2\pi \). But \( G^e(x) \) is an even function while \( G^o(x) \) is an odd function. Because it is an even periodic function, \( G^e(x) \) has the Fourier series expansion
\[ G^e(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \left( \frac{k\pi x}{\ell} \right) \]
with
\[ a_k = \frac{2}{\pi} \int_{0}^{\pi} 1 \cos \left( \frac{k\pi x}{\ell} \right) \, dx = \begin{cases} 2 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases} \]
That is,
\[ G^e(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \left( \frac{k\pi x}{\ell} \right) = 1 \quad \text{(surprise!)} \]

Because it is an odd periodic function, \( G^o(x) \) has the Fourier series expansion
\[ G^o(x) = \sum_{k=1}^{\infty} b_k \sin \left( \frac{k\pi x}{\ell} \right) \]
with
\[ b_k = \frac{2}{\pi} \int_{0}^{\pi} 1 \sin \left( \frac{k\pi x}{\ell} \right) \, dx = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{4}{k\pi} & \text{if } k \text{ is odd} \end{cases} \]
That is
\[ G^o(x) = \sum_{k=1}^{\infty} \frac{4}{k\pi} \sin \left( \frac{k\pi x}{\ell} \right) \]
For all \( 0 < x < \pi \) we have both
\[ g(x) = G^e(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \left( \frac{k\pi x}{\ell} \right) = 1 \]
\[ g(x) = G^o(x) = \sum_{k=1}^{\infty} b_k \sin \left( \frac{k\pi x}{\ell} \right) = \sum_{k=1, k \text{ odd}}^{\infty} \frac{4}{k\pi} \sin \left( \frac{k\pi x}{\ell} \right) \]

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