A Lightning Fast Review of Eigenvalues and Eigenvectors

Let $A$ be an $n \times n$ matrix. Then, by definition, $\vec{v}$ is an eigenvector of $A$ with eigenvalue $\lambda$ if and only if

(i) $\vec{v} \neq \vec{0}$
(ii) $A\vec{v} = \lambda \vec{v}$

Fix $\lambda$ for a moment. Then $(A - \lambda I)\vec{v} = \vec{0}$ has a nonzero solution $\vec{v}$ if and only if the matrix $A - \lambda I$ fails to have an inverse. This, in turn, is the case if and only if the matrix has determinant zero. Hence the eigenvalues of $A$ are determined by

$$
\det(A - \lambda I) = 0
$$

This determinant is a polynomial in $\lambda$ of degree $n$. Consequently, it has precisely $n$ roots, counting multiplicity. Given an eigenvalue $\lambda$ of $A$ the corresponding eigenvectors are found by solving

$$(A - \lambda I)\vec{v} = \vec{0}$$

Each distinct eigenvector has at least one eigenvector. If $\lambda$ has multiplicity $m$ then $\lambda$ may have form 1 to $m$ linearly independent eigenvectors.

Example 1 $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$$
0 = \det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 2 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)
$$

The eigenvalues of $A$ are $\lambda = -1, 2$.

For $\lambda = -1$, \begin{pmatrix} 1 - \lambda & 2 \\ 1 & -\lambda \end{pmatrix} \vec{v} = 0 \implies \begin{pmatrix} 2 \\ 1 \end{pmatrix} \vec{v} = \vec{0} \implies \vec{v} = \text{const} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For $\lambda = 2$, \begin{pmatrix} 1 - \lambda & 2 \\ 1 & -\lambda \end{pmatrix} \vec{v} = 0 \implies \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \vec{v} = \vec{0} \implies \vec{v} = \text{const} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Example 2 $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$
0 = \det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 1 & 1 \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^3
$$

The eigenvalues of $A$ are $\lambda = 1, 1, 1$. For $\lambda = 1$

$$
\begin{pmatrix} 1 - \lambda & 1 & 1 \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix} \vec{v} = 0 \implies \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0} \implies v_2 + v_3 = 0
$$
which has general solution \( v_1 = \alpha \), arbitrary, \( v_3 = \beta \), arbitrary, and \( v_2 = -\beta \) or

\[
\vec{v} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}
\]

There are two linearly independent eigenvectors of eigenvalue 1, which can be chosen to be \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \), \( \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \).

**Remarks**

1) If \( A \) is \( 2 \times 2 \), then \( \det(A - \lambda I) = \lambda^2 - (\text{tr} A)\lambda + \det A \).

If \( A \) is \( 3 \times 3 \), then \( \det(A - \lambda I) = -\lambda^3 + (\text{tr} A)\lambda^2 - \left( \sum_{i=1}^{3} \det M_i \right) \lambda + \det A \).

Here \( \text{tr} A = \sum_{i=1}^{3} A_{ii} \) is the trace of \( A \) and \( M_i \) is the \( 2 \times 2 \) matrix gotten by deleting the \( i^{\text{th}} \) row and column from \( A \).

2) If \( A \) is triangular (i.e. all entries on one side of the diagonal of \( A \) are zero) then the eigenvalues of \( A \) are just the diagonal entries of \( A \).

3) If \( A_{ij} = A_{ji} \) for all \( 1 \leq i, j \leq n \), (such matrices are called symmetric, or Hermitian or self-adjoint) then all of the eigenvalues of \( A \) are real and \( A \) has \( n \) linearly independent eigenvectors (even if \( A \) has multiple eigenvalues).

4) Suppose that the \( n \times n \) matrix \( A \) has \( n \) linearly independent eigenvectors \( \vec{v}_i \) with corresponding eigenvalues \( \lambda_i \). Define \( V = (\vec{v}_1, \cdots, \vec{v}_n) \) (i.e. \( V \) is the matrix whose \( j^{\text{th}} \) column is \( \vec{v}_j \)) and define \( \Lambda \) to be the diagonal matrix whose \( (j, j) \) entry is \( \lambda_j \). Then

\[
A \vec{v}_j = \lambda_j \vec{v}_j, \ 1 \leq j \leq n
\]

is equivalent to

\[
AV = A (\vec{v}_1, \cdots, \vec{v}_n) = (A \vec{v}_1, \cdots, A \vec{v}_n) = (\lambda_1 \vec{v}_1, \cdots, \lambda_n \vec{v}_n) = V \Lambda
\]

by the definition of matrix multiplication. Hence

\[
A = V \Lambda V^{-1} \quad V^{-1} AV = \Lambda
\]