

A Lightning Fast Review of Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. Then, by **definition**, \vec{v} is an eigenvector of A with eigenvalue λ if and only if

- (i) $\vec{v} \neq \vec{0}$
- (ii) $A\vec{v} = \lambda\vec{v}$

Fix λ for a moment. Then $(A - \lambda I)\vec{v} = \vec{0}$ has a nonzero solution \vec{v} if and only if the matrix $A - \lambda I$ fails to have an inverse. This, in turn, is the case if and only if the matrix has determinant zero. Hence the eigenvalues of A are determined by

$$\det(A - \lambda I) = 0$$

This determinant is a polynomial in λ of degree n . Consequently, it has precisely n roots, counting multiplicity. Given an eigenvalue λ of A the corresponding eigenvectors are found by solving

$$(A - \lambda I)\vec{v} = \vec{0}$$

Each distinct eigenvalue has at least one eigenvector. If λ has multiplicity m then λ may have from 1 to m linearly independent eigenvectors.

Example 1 $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 2 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$$

The eigenvalues of A are $\lambda = -1, 2$.

$$\text{For } \lambda = -1, \quad \begin{pmatrix} 1 - \lambda & 2 \\ 1 & -\lambda \end{pmatrix} \vec{v} = 0 \implies \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \vec{v} = \vec{0} \implies \vec{v} = \text{const} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{For } \lambda = 2, \quad \begin{pmatrix} 1 - \lambda & 2 \\ 1 & -\lambda \end{pmatrix} \vec{v} = 0 \implies \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \vec{v} = \vec{0} \implies \vec{v} = \text{const} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Example 2 $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 1 & 1 \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^3$$

The eigenvalues of A are $\lambda = 1, 1, 1$. For $\lambda = 1$

$$\begin{pmatrix} 1 - \lambda & 1 & 1 \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix} \vec{v} = 0 \implies \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \vec{0} \implies v_2 + v_3 = 0$$

which has general solution $v_1 = \alpha$, arbitrary, $v_3 = \beta$, arbitrary, and $v_2 = -\beta$ or

$$\vec{v} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

There are two linearly independent eigenvectors of eigenvalue 1, which can be chosen to be $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

Remarks

- 1) If A is 2×2 , then $\det(A - \lambda I) = \lambda^2 - (\text{tr } A)\lambda + \det A$.
If A is 3×3 , then $\det(A - \lambda I) = -\lambda^3 + (\text{tr } A)\lambda^2 - \left(\sum_{i=1}^3 \det M_i\right)\lambda + \det A$.
Here $\text{tr } A = \sum_{i=1}^3 A_{ii}$ is the trace of A and M_i is the 2×2 matrix gotten by deleting the i^{th} row and column from A .
- 2) If A is triangular (i.e. all entries on one side of the diagonal of A are zero) then the eigenvalues of A are just the diagonal entries of A .
- 3) If $A_{ij} = \overline{A_{ji}}$ for all $1 \leq i, j \leq n$, (such matrices are called symmetric, or Hermitian or self-adjoint) then all of the eigenvalues of A are real and A has n linearly independent eigenvectors (even if A has multiple eigenvalues).
- 4) Suppose that the $n \times n$ matrix A has n linearly independent eigenvectors \vec{v}_i with corresponding eigenvalues λ_i . Define $V = (\vec{v}_1, \dots, \vec{v}_n)$ (i.e. V is the matrix whose j^{th} column is \vec{v}_j) and define Λ to be the diagonal matrix whose (j, j) entry is λ_j . Then

$$A\vec{v}_j = \lambda_j\vec{v}_j, \quad 1 \leq j \leq n$$

is equivalent to

$$AV = A(\vec{v}_1, \dots, \vec{v}_n) = (A\vec{v}_1, \dots, A\vec{v}_n) = (\lambda_1\vec{v}_1, \dots, \lambda_n\vec{v}_n) = V\Lambda$$

by the definition of matrix multiplication. Hence

$$A = V\Lambda V^{-1} \quad V^{-1}AV = \Lambda$$

Exercises Find the eigenvalues and eigenvectors of

$$a) \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \quad b) \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad c) \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix}$$

$$d) \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \quad e) \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \quad f) \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

$$g) \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix}$$