Math 227 Problem Set IX Solutions

1. Evaluate, both by direct integration and by Stokes' Theorem, $\oint_C (z\,dx + x\,dy + y\,dz)$ where C is the circle $x+y+z=0$, $x^2+y^2+z^2=1$. Orient C so that its projection on the xy–plane is counterclockwise.

Solution. *Direct integration:* The plane $x+y+z=0$ passes through the centre, $(0,0,0)$, of the sphere $x^2+y^2+z^2=1$, so the circle C is centred on $(0,0,0)$ and has radius 1. It lies in a plane with unit (upward pointing) normal vector $\hat{\mathbf{k}}' = \frac{1}{\sqrt{3}}(1,1,1)$. The point $\frac{1}{\sqrt{2}}(1,-1,0)$ is on C, so the vector $\hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{2}}(1,-1,0)$ lies in the plane of C. Another vector which lies in the plane of C, but which is perpendciular to $\hat{\boldsymbol{\imath}}'$ is $\hat{\boldsymbol{\jmath}}' = \hat{\mathbf{k}}' \times \hat{\boldsymbol{\imath}}' = \frac{1}{\sqrt{6}}(1,1,-2)$. Hence we may parametrize the curve using

$$\mathbf{r}(\theta) = \cos\theta\,\hat{\boldsymbol{\imath}}' + \sin\theta\,\hat{\boldsymbol{\jmath}}'$$

which gives

$$\begin{cases} x(\theta) = \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{6}}\sin\theta \\ y(\theta) = -\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{6}}\sin\theta \\ z(\theta) = -\frac{2}{\sqrt{6}}\sin\theta \end{cases} \qquad \text{and} \qquad \begin{cases} x'(\theta) = -\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{6}}\cos\theta \\ y'(\theta) = \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{6}}\cos\theta \\ z'(\theta) = -\frac{2}{\sqrt{6}}\cos\theta \end{cases}$$

The check that motion, projected on the xy–plane, is counterclockwise, observe that at $\theta=0$, $(x,y) = \frac{1}{\sqrt{2}}(1,-1)$ (in the fourth quadrant) and $\left(\frac{dx}{d\theta}, \frac{dy}{d\theta}\right) = \frac{1}{\sqrt{6}}(1,1)$, which is up and to the right. The integral is of the form $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = z\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}} + y\hat{\mathbf{k}}$ and C is curve parametrized above. Hence

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}\big(\mathbf{r}(\theta)\big)\cdot \mathbf{r}'(\theta)\ d\theta \\
&= \int_0^{2\pi} \Big[-\tfrac{2}{\sqrt{6}}\sin\theta\big(-\tfrac{1}{\sqrt{2}}\sin\theta + \tfrac{1}{\sqrt{6}}\cos\theta\big) \\
&\qquad + \big(\tfrac{1}{\sqrt{2}}\cos\theta + \tfrac{1}{\sqrt{6}}\sin\theta\big)\big(\tfrac{1}{\sqrt{2}}\sin\theta + \tfrac{1}{\sqrt{6}}\cos\theta\big) \\
&\qquad + \big(-\tfrac{1}{\sqrt{2}}\cos\theta + \tfrac{1}{\sqrt{6}}\sin\theta\big)\big(-\tfrac{2}{\sqrt{6}}\cos\theta\big)\Big]\ d\theta \\
&= \int_0^{2\pi} \Big[\tfrac{3}{\sqrt{12}}\cos^2\theta + \tfrac{3}{\sqrt{12}}\sin^2\theta + \big(-\tfrac{2}{6} + \tfrac{1}{6} + \tfrac{1}{2} - \tfrac{2}{6}\big)\sin\theta\cos\theta\Big]\ d\theta \\
&= \tfrac{3}{\sqrt{12}}2\pi = \boxed{\sqrt{3}\pi}
\end{aligned}$$

Stokes' Theorem: Choose as S the portion of the plane $x+y+z=0$ interior to the sphere. Then $\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})$ and $\boldsymbol{\nabla}\times\mathbf{F} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$ so, by Stokes' Theorem,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\ dS = \iint_S (\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})\cdot\tfrac{1}{\sqrt{3}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})\ dS = \sqrt{3}\iint_S dS = \boxed{\sqrt{3}\pi}$$

since S is a circle of radius 1.

2. Let C be the intersection of $x+2y-z=7$ and $x^2-2x+4y^2=15$. The curve C is oriented counterclockwise when viewed from high on the z–axis. Let

$$\mathbf{F} = \big(e^{x^2} + yz\big)\hat{\boldsymbol{\imath}} + \big(\cos(y^2) - x^2\big)\hat{\boldsymbol{\jmath}} + \big(\sin(z^2) + xy\big)\hat{\mathbf{k}}$$

Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

Solution. We apply Stokes' Theorem with S being the part of the plane $x+2y-z=7$ that is inside the ellipse $x^2-2x+4y^2=15$ (which can also be written $(x-1)^2+4y^2=16$.) We can parametrize S by $z=-7+x+2y$, with $(x-1)^2+4y^2\le 16$, so that $\hat{\mathbf{n}}dS=(-1,-2,1)\,dxdy$ (upward normal). So, by Stokes' Theorem and the observation that $\boldsymbol{\nabla}\times\mathbf{F}=x\hat{\boldsymbol{\imath}}-(z+2x)\hat{\mathbf{k}}$, and using R to denote the interior of the ellipse $(x-1)^2+4y^2=16$,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS\\ &= \iint_R [x\hat{\boldsymbol{\imath}}-(z+2x)\hat{\mathbf{k}}]\big|_{z=-7+x+2y}\cdot(-1,-2,1)\,dx\,dy\\ &= \iint_R [7-4x-2y]\,dx\,dy\\ &= (\text{area of ellipse with semi–axes } a=4,\ b=2)[7-4\bar{x}-2\bar{y}]\\ &= \pi\times 4\times 2[7-4\times 1-2\times 0] = \boxed{24\pi}\end{aligned}$$

Here we have used that, for example, the average value of x over R is $\bar{x}=\iint_R x\,dx\,dy/\iint_R dx\,dy$.

3. Consider $\iint_S(\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS$ where S is the portion of the sphere $x^2+y^2+z^2=1$ that obeys $x+y+z\ge 1$, $\hat{\mathbf{n}}$ is the upward pointing normal to the sphere and $\mathbf{F}=(y-z)\hat{\boldsymbol{\imath}}+(z-x)\hat{\boldsymbol{\jmath}}+(x-y)\hat{\mathbf{k}}$. Find another surface S' with the property that $\iint_S(\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS=\iint_{S'}(\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS$ and evaluate $\iint_{S'}(\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS$.

Solution. Let S' be the portion of $x+y+z=1$ that is inside the sphere $x^2+y^2+z^2=1$. Then $\partial S=\partial S'$, so, by Stokes' Theorem, (with $\hat{\mathbf{n}}$ always the upward pointing normal)

$$\iint_{S'}(\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS=\oint_{\partial S'}\mathbf{F}\cdot d\mathbf{r}=\oint_{\partial S}\mathbf{F}\cdot d\mathbf{r}=\iint_S(\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS$$

As $\boldsymbol{\nabla}\times\mathbf{F}=-2(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}})$ and, on S', $\hat{\mathbf{n}}=\frac{1}{\sqrt{3}}(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}})$

$$\iint_{S'}(\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS=\iint_{S'}\big(-2\sqrt{3}\big)\,dS=-2\sqrt{3}\times\text{Area}(S')$$

S' is a circular disk. It's center (x_c,y_c,z_c) has to obey $x_c+y_c+z_c=1$. By symmetry, $x_c=y_c=z_c$, so $x_c=y_c=z_c=\frac{1}{3}$. Any point, like $(0,0,1)$, which satisfies both $x+y+z=1$ and $x^2+y^2+z^2=1$ is on the boundary of S'. So the radius of S' is $\big|\big(\frac{1}{3},\frac{1}{3},\frac{1}{3}\big)-(0,0,1)\big|=\big|\big(\frac{1}{3},\frac{1}{3},-\frac{2}{3}\big)\big|=\sqrt{\frac{2}{3}}$. So the area of S' is $\frac{2}{3}\pi$ and

$$\iint_{S'}(\boldsymbol{\nabla}\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS=-2\sqrt{3}\times\text{Area}(S')=\boxed{-\tfrac{4}{\sqrt{3}}\pi}$$

4. Verify the identity $\oint_C\phi\boldsymbol{\nabla}\psi\cdot d\mathbf{r}=-\oint_C\psi\boldsymbol{\nabla}\phi\cdot d\mathbf{r}$ for any continuously differentiable scalar fields ϕ and ψ and curve C that is the boundary of a piecewise smooth surface.

Solution. Suppose that $C=\partial S$. Then, by Stokes' Theorem

$$\oint_C[\phi\boldsymbol{\nabla}\psi+\psi\boldsymbol{\nabla}\phi]\cdot d\mathbf{r}=\iint_S\boldsymbol{\nabla}\times[\phi\boldsymbol{\nabla}\psi+\psi\boldsymbol{\nabla}\phi]\cdot\hat{\mathbf{n}}\,dS$$

But

$$\begin{aligned}\boldsymbol{\nabla}\times[\phi\boldsymbol{\nabla}\psi+\psi\boldsymbol{\nabla}\phi] &= \boldsymbol{\nabla}\phi\times\boldsymbol{\nabla}\psi+\phi\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi)+\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\phi+\psi\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi)\\ &= \boldsymbol{\nabla}\phi\times\boldsymbol{\nabla}\psi+\boldsymbol{\nabla}\psi\times\boldsymbol{\nabla}\phi \qquad \text{since } \phi\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi)=\psi\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi=0\\ &= 0\end{aligned}$$

so $\oint_C [\phi\boldsymbol{\nabla}\psi + \psi\boldsymbol{\nabla}\phi]\cdot d\mathbf{r} = 0$.

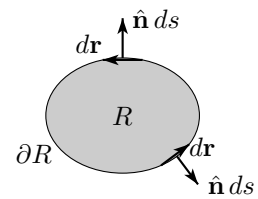
5. Use Green's Theorem to show that

$$\iint_R \boldsymbol{\nabla}\cdot\mathbf{F}\ dA = \oint_{\partial R} \mathbf{F}\cdot\hat{\mathbf{n}}\, ds$$

where ∂R is the boundary of the plane domain R, with the usual orientation, $\mathbf{F} = F_1\hat{\boldsymbol{\imath}} + F_2\hat{\boldsymbol{\jmath}}$, $\hat{\mathbf{n}}$ is the outward normal to ∂R and s is the arclength along ∂R.

Solution. If $d\mathbf{r} = dx\,\hat{\boldsymbol{\imath}} + dy\,\hat{\boldsymbol{\jmath}}$ is an infinitesmal piece of ∂R, then $\hat{\mathbf{n}}\,ds = d\mathbf{r}\times\hat{\mathbf{k}} = dy\,\hat{\boldsymbol{\imath}} - dx\,\hat{\boldsymbol{\jmath}}$. (Note that $d\mathbf{r}\times\hat{\mathbf{k}}$ has the same length as $d\mathbf{r}$, lies in the xy–plane and is perpendicular to $d\mathbf{r}$. Use the right hand rule to check that $d\mathbf{r}\times\hat{\mathbf{k}}$ is $\hat{\mathbf{n}}\,ds$ rather than $-\hat{\mathbf{n}}\,ds$.) So, by Green's Theorem

$$\oint_{\partial R} \mathbf{F}\cdot\hat{\mathbf{n}}\,ds = \oint_{\partial R}[-F_2\,dx + F_1\,dy] = \iint_R \left[\tfrac{\partial}{\partial x}F_1 - \tfrac{\partial}{\partial y}(-F_2)\right]\ dA = \iint_R \boldsymbol{\nabla}\cdot\mathbf{F}\ dA$$



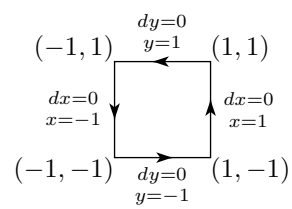
6. Integrate $\frac{1}{2\pi}\oint_C \frac{x\,dy - y\,dx}{x^2+y^2}$ counterclockwise around
   (a) the circle $x^2 + y^2 = a^2$
   (b) the boundary of the square with vertices $(-1,-1)$, $(-1,1)$, $(1,1)$ and $(1,-1)$
   (c) the boundary of the region $1 \le x^2 + y^2 \le 2,\ y \ge 0$

Solution. (a) Parametrize the circle by $x = a\cos\theta,\ y = a\sin\theta$, $0 \le \theta \le 2\pi$. Then $dx = -a\sin\theta\,d\theta$ and $dy = a\cos\theta\,d\theta$ so

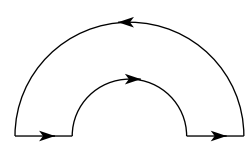
$$\frac{1}{2\pi}\oint_C \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2\cos^2\theta\,d\theta + a^2\sin^2\theta\,d\theta}{a^2\cos^2\theta + a^2\sin^2\theta} = \frac{1}{2\pi}\int_0^{2\pi} d\theta = \boxed{1}$$

(b) From the figure on the right,

$$\frac{1}{2\pi}\oint_C \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{1}{2\pi}\int_{-1}^{1}\frac{dx}{x^2+1} + \frac{1}{2\pi}\int_{-1}^{1}\frac{dy}{1+y^2} + \frac{1}{2\pi}\int_{1}^{-1}\frac{-dx}{x^2+1} + \frac{1}{2\pi}\int_{1}^{-1}\frac{-dy}{1+y^2}$$
$$= 4\frac{1}{2\pi}\tan^{-1}x\Big|_{-1}^{1} = \frac{2}{\pi}\left[\frac{\pi}{4} + \frac{\pi}{4}\right] = \boxed{1}$$



(c) The outer semicircle gives $\frac{1}{2}$, as in part (a). The inner semicircle gives $-\frac{1}{2}$, as in part (a). The two flat pieces each give zero, since on them $y = 0$ and $dy = 0$. So $\boxed{\frac{1}{2\pi}\oint_C \frac{x\,dy - y\,dx}{x^2+y^2} = 0}$.



7. Show that

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$$

for all $(x,y) \ne (0,0)$. Discuss the connection between this result and the results of the last question.

Solution.

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$
$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-(x^2+y^2)-(-y)(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

The two are well–defined and equal everywhere except at the origin $(0,0)$. Were it not for the singularity at $(0,0)$, the vector field of the last problem would be conservative and the integral $\int \vec{F}\cdot d\vec{r}$ around any

closed curve would be zero. But as we saw in parts (a) and (b) of the last problem, this is not the case. On the other hand, by Green's Theorem (or Stokes' Theorem), the integral around the boundary of any region that does not contain $(0,0)$ is zero, as happened in part (c).

8. Find a continuously differentiable simple, closed, counterclockwise oriented curve, C, in the xy–plane for the which the value of the line integral $\oint_C (y^3 - y)\,dx - 2x^3\,dy$ is a maximum among all C^1 simple, closed, counterclockwise oriented curves. "Simple" means that the curve does not intersect itself.

   Solution. By Green's Theorem

$$\oint_C (y^3 - y)\,dx - 2x^3\,dy = \iint_R \left[\tfrac{\partial}{\partial x}(-2x^3) - \tfrac{\partial}{\partial y}(y^3 - y)\right]\,dx\,dy = \iint_R \left[1 - 6x^2 - 3y^2\right]\,dx\,dy$$

   where R is the region in the xy–plane whose boundary is C. Observe that the integrand $1 - 6x^2 - 3y^2$ is positive in the elliptical region $6x^2 + 3y^2 \le 1$ and negative outside of it. To maximize the integral $\iint_R \left[1-6x^2-3y^2\right]\,dx\,dy$ we should choose R to contain all points (x,y) with the integrand $1-6x^2-3y^2 \ge 0$ and to exclude all points (x,y) with the integrand $1 - 6x^2 - 3y^2 < 0$. So we choose

$$R = \left\{\ (x,y)\ \middle|\ 6x^2 + 3y^2 \le 1\ \right\}$$

   The corresponding C is $\boxed{6x^2 + 3y^2 = 1}$.