Math 227 Problem Set II Solutions

1. Find the curvature of the plane curve \( y = e^x \).

**Solution.** This curve has equation \( y = f(x) \) with \( f'(x) = e^x \). So
\[
\kappa(x) = \frac{f''(x)}{[1 + f'(x)^2]^{3/2}} = \frac{e^x}{[1 + e^{2x}]^{3/2}}
\]

2. Find the minimum and maximum values for the curvature of the ellipse \( x(t) = a \cos t \), \( y(t) = b \sin t \). Here \( a > b > 0 \).

**Solution.** For the given ellipse
\[
\mathbf{r}(t) = a \cos t \mathbf{i} + b \sin t \mathbf{j}
\]
\[
\mathbf{v}(t) = -a \sin t \mathbf{i} + b \cos t \mathbf{j}, \quad |\mathbf{v}(t)| = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}
\]
\[
\mathbf{a}(t) = -a \cos t \mathbf{i} - b \sin t \mathbf{j}, \quad \mathbf{v}(t) \times \mathbf{a}(t) = ab \mathbf{k}
\]
\[
\kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{ab}{[a^2 \sin^2 t + b^2 \cos^2 t]^{3/2}}
\]

Hence the maximum (minimum) curvature is achieved when the denominator is a minimum (maximum) which is the case when \( \sin t = 0 \) (cos \( t = 0 \)). So \( \kappa_{\text{max}} = \frac{a}{b^2} \) and \( \kappa_{\text{min}} = \frac{b}{a^2} \).

3. Let \( r = f(\theta) \) be the equation of a plane curve in polar coordinates. Find the curvature of this curve at a general point \( \theta \).

**Solution.** Think of \( \theta \) as a time parameter. Then the given curve has
\[
\mathbf{r}(\theta) = f(\theta) \left[ \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \right]
\]
\[
\mathbf{v}(\theta) = f'(\theta) \left[ \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \right] + f(\theta) \left[ -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \right]
\]
\[
|\mathbf{v}(\theta)| = \sqrt{f'(\theta)^2 + f(\theta)^2}
\]
\[
\mathbf{a}(\theta) = \left\{ f''(\theta) - f(\theta) \right\} \left[ \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \right] + 2f'(\theta) \left[ -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \right]
\]
\[
\mathbf{v}(\theta) \times \mathbf{a}(\theta) = \left\{ 2f'(\theta)^2 - f(\theta) \left[ f''(\theta) - f(\theta) \right] \right\} \mathbf{k}
\]
\[
\kappa(\theta) = \frac{|\mathbf{v}(\theta) \times \mathbf{a}(\theta)|}{|\mathbf{v}(\theta)|^3} = \frac{|f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta)|}{[f'(\theta)^2 + f(\theta)^2]^{3/2}}
\]

The computation of the cross product \( \mathbf{v}(\theta) \times \mathbf{a}(\theta) \) (and also of \( |\mathbf{v}(\theta)| \)) has been facilitated by the observation that \( \mathbf{e}_\theta(\theta) = \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \) and \( \mathbf{e}_\phi(\theta) = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \) are mutually perpendicular unit vectors obeying \( \mathbf{e}_\tau(\theta) \times \mathbf{e}_\theta(\theta) = \mathbf{k} \) and \( \mathbf{e}_\tau(\theta) \times \mathbf{e}_\phi(\theta) = \mathbf{e}_\phi(\theta) \times \mathbf{e}_\theta(\theta) = 0 \).

4. Find the curvature of the cardioid \( r = a(1 - \cos \theta) \).

**Solution.** By the previous problem with
\[
f(\theta) = a(1 - \cos \theta) \quad f'(\theta) = a \sin \theta \quad f''(\theta) = a \cos \theta
\]
we have
\[
\kappa(\theta) = \frac{|f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta)|}{[f'(\theta)^2 + f(\theta)^2]^{3/2}}
\]
\[
= \frac{|a^2 - 2a^2 \cos \theta + a^2 \cos^2 \theta + a^2 \sin^2 \theta - a^2 \cos \theta + a^2 \cos^2 \theta|}{[a^2 - 2a^2 \cos \theta + a^2 \cos^2 \theta + a^2 \sin^2 \theta]^{3/2}}
\]
\[
= \frac{3a^2 - 3a^2 \cos \theta}{[2a^2 - 2a^2 \cos \theta]^{3/2}} = \frac{3}{2^{3/2}a\sqrt{1 - \cos \theta}} = \frac{3}{2\sqrt{2a^2(1 - \cos \theta)}}
\]
5. Find the unit tangent, unit normal and binormal vectors and the curvature and torsion of the curve

\[ \mathbf{r}(t) = t \mathbf{i} + \frac{t^2}{2} \mathbf{j} + \frac{t^3}{3} \mathbf{k} \]

Solution. For the specified curve

\[ \mathbf{v}(t) = \mathbf{i} + t \mathbf{j} + t^2 \mathbf{k} \]
\[ \mathbf{a}(t) = 3 \mathbf{j} + 2t \mathbf{k} \]
\[ \mathbf{v}(\theta) \times \mathbf{a}(\theta) = t^2 \mathbf{i} - 2t \mathbf{j} + \mathbf{k} \]
\[ \mathbf{a}'(t) = 2 \mathbf{k} \]

From this, we read off

\[
\begin{align*}
\mathbf{T}(t) &= \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{i} + t \mathbf{j} + t^2 \mathbf{k}}{\sqrt{1 + t^2 + t^4}} \\
\kappa(t) &= \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{\sqrt{1 + 4t^2 + t^4}}{|1 + t^2 + t^4|^{3/2}} \\
\mathbf{B}(t) &= \frac{\mathbf{v}(t) \times \mathbf{a}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|} = \frac{t^2 \mathbf{i} - 2t \mathbf{j} + \mathbf{k}}{\sqrt{1 + 4t^2 + t^4}} \\
\mathbf{N}(t) &= \mathbf{B}(t) \times \mathbf{T}(t) = \frac{-(t + 2t^3) \mathbf{i} + (1 - t^4) \mathbf{j} + (2t + 3t^2) \mathbf{k}}{\sqrt{1 + t^2 + t^4} \sqrt{1 + 4t^2 + t^4}} \\
\tau(t) &= \frac{(\mathbf{v}(t) \times \mathbf{a}(t)) \cdot \mathbf{a}'(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|^2} = \frac{2}{1 + 4t^2 + t^4}
\end{align*}
\]

6. (a) Show that if a curve has curvature \( \kappa(s) = 0 \) for all \( s \), then the curve is a straight line.
(b) Show that if a curve has torsion \( \tau(s) = 0 \) for all \( s \), then the curve lies in a plane.
(c) Show that if a curve has curvature \( \kappa(s) = \kappa_0 \), a strictly positive constant, and torsion \( \tau(s) = 0 \) for all \( s \), then the curve is a circle.

Solution. (a) If \( \kappa(s) \equiv 0 \), then \( \frac{d\mathbf{P}}{ds} = \kappa(s) \mathbf{N}(s) \equiv 0 \) so that \( \mathbf{T} \) is a constant. As a result \( \frac{d\mathbf{T}}{ds}(s) = \mathbf{T} \) and \( \mathbf{r}(s) = s \mathbf{T} + \mathbf{r}(0) \) so that the curve is the straight line with direction vector \( \mathbf{T} \) that passes through \( \mathbf{r}(0) \).
(b) If \( \tau(s) \equiv 0 \), then \( \frac{d\mathbf{B}}{ds} = -\mathbf{N}(s) \mathbf{B} \equiv 0 \) so that \( \mathbf{B} \) is a constant. As \( \mathbf{T}(s) \perp \mathbf{B} \),

\[
\frac{d}{ds}(\mathbf{r}(s) - \mathbf{r}(0)) \cdot \mathbf{B} = \mathbf{T}(s) \cdot \mathbf{B} = 0
\]

and \( (\mathbf{r}(s) - \mathbf{r}(0)) \cdot \mathbf{B} \) must be a constant. The constant must be zero (set \( s = 0 \)), so \( (\mathbf{r}(s) - \mathbf{r}(0)) \cdot \mathbf{B} = 0 \) and \( \mathbf{r}(s) \) always lies in the plane through \( \mathbf{r}(0) \) with normal vector \( \mathbf{B} \).
(c) Parametrize the curve by arc length. Define the “centre of curvature” at \( s \) by

\[
\mathbf{r}_c(s) = \mathbf{r}(s) + \frac{1}{\kappa(s)} \mathbf{N}(s)
\]

Since \( \kappa(s) = \kappa_0 \) is a constant and \( \tau(s) \equiv 0 \),

\[
\frac{d}{ds} \mathbf{r}_c(s) = \mathbf{T}(s) + \frac{1}{\kappa_0} [\tau(s) \mathbf{B} - \kappa(s) \mathbf{T}] = \mathbf{T}(s) + \frac{1}{\kappa_0} [0 \mathbf{B} - \kappa_0 \mathbf{T}] = 0
\]

Thus \( \mathbf{r}_c(s) = \mathbf{r}_c \) is a constant and \( |\mathbf{r}(s) - \mathbf{r}_c| = \frac{1}{\kappa_0} \) lies on the sphere of radius \( \frac{1}{\kappa_0} \) centred on \( \mathbf{r}_c \). Since \( \tau(s) \equiv 0 \), the curve also lies on a plane, so it is a circle.
7. Solve the initial value problem
\[ \frac{dr}{dt} = \mathbf{k} \times \mathbf{r} \quad \mathbf{r}(0) = \mathbf{i} + \mathbf{k} \]
Describe the curve \( \mathbf{r} = \mathbf{r}(t) \).

**Solution.** The velocity vector \( \frac{dr}{dt} = \mathbf{k} \times \mathbf{r} \) is always perpendicular to \( \mathbf{k} \). Consequently, the curve lies in a plane perpendicular to \( \mathbf{k} \). Furthermore, the velocity vector is always perpendicular to \( \mathbf{r} \). So the curve lies on a sphere. (See problem #7 of Problem Set I.) The intersection of a sphere with a plane is a circle, so the path is a circle. We are told that \( \mathbf{r}(0) = \mathbf{i} + \mathbf{k} \). Guess \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \mathbf{k} \). As

\[
\left. \cos t \mathbf{i} + \sin t \mathbf{j} + \mathbf{k} \right|_{t=0} = \mathbf{i} + \mathbf{k}
\]
\[
\frac{d}{dt} \left[ \cos t \mathbf{i} + \sin t \mathbf{j} + \mathbf{k} \right] = -\sin t \mathbf{i} + \cos t \mathbf{j}
\]
\[
\mathbf{k} \times \left[ \cos t \mathbf{i} + \sin t \mathbf{j} + \mathbf{k} \right] = \cos t \mathbf{j} - \sin t \mathbf{i}
\]
the guess satisfies both the required differential equation and the required initial condition.

8. Find the curvature \( \kappa \) as a function of arclength \( s \) (measured from \((0,0)\)) for the curve
\[
x(\theta) = \int_0^\theta \cos \left( \frac{1}{2} \pi t^2 \right) dt \quad y(\theta) = \int_0^\theta \sin \left( \frac{1}{2} \pi t^2 \right) dt
\]

**Solution.** The velocity vector is
\[
\mathbf{v}(\theta) = x'(\theta) \mathbf{i} + y'(\theta) \mathbf{j} = \cos \left( \frac{1}{2} \pi \theta^2 \right) \mathbf{i} + \sin \left( \frac{1}{2} \pi \theta^2 \right) \mathbf{j}
\]
Consequently the speed
\[
\frac{d}{dt}(\theta) = |\mathbf{v}(\theta)| = 1 \quad \implies \quad s(\theta) = \theta + s(0)
\]
Since \( s(\theta) \) is zero when \( \theta = 0 \), we have \( s(\theta) = \theta \) and hence \( \mathbf{T}(s) = \mathbf{v}(s) = \cos \left( \frac{1}{2} \pi s^2 \right) \mathbf{i} + \sin \left( \frac{1}{2} \pi s^2 \right) \mathbf{j} \) so that
\[
\kappa(s) = \left| \frac{d}{ds} \mathbf{T}(s) \right| = \left| -\pi s \sin \left( \frac{1}{2} \pi s^2 \right) \mathbf{i} + \pi s \cos \left( \frac{1}{2} \pi s^2 \right) \mathbf{j} \right| = \pi s
\]

9. A frictionless roller–coaster track has the form of one turn of the circular helix with parametrization \( (a \cos \theta, a \sin \theta, b \theta) \). A car leaves the point where \( \theta = 2\pi \) with zero velocity and moves under gravity to the point where \( \theta = 0 \). By Newton’s law of motion, the position \( \mathbf{r}(t) \) of the car at time \( t \) obeys
\[
m \mathbf{r}''(t) = \mathbf{N}(\mathbf{r}(t)) - mg \mathbf{k}
\]
Here \( m \) is the mass of the car, \( g \) is a constant, \( -mg \mathbf{k} \) is the force due to gravity and \( \mathbf{N}(\mathbf{r}(t)) \) is the force that the roller–coaster track applies to the car to keep the car on the track. Since the track is frictionless, \( \mathbf{N}(\mathbf{r}(t)) \) is always perpendicular to \( \mathbf{v}(t) = \frac{d}{dt}(\theta) \).

(a) Prove that \( E(t) = \frac{1}{2} m |\mathbf{v}(t)|^2 + m g \mathbf{r}(t) \cdot \mathbf{k} \) is a constant, independent of \( t \). (This is called “conservation of energy”.)

(b) Prove that the speed \( |\mathbf{v}| \) at the point \( \theta \) obeys \( |\mathbf{v}|^2 = 2gb(2\pi - \theta) \).

(c) Find the time it takes to reach \( \theta = 0 \).

**Solution.** (a) By Newton’s law of motion
\[
E'(t) = \frac{d}{dt} \left[ \frac{1}{2} m |\mathbf{v}(t)|^2 + m g \mathbf{r}(t) \cdot \mathbf{k} \right] = m \mathbf{v}(t) \cdot \mathbf{v}'(t) + m g \mathbf{v}(t) \cdot \mathbf{k} = \mathbf{v}(t) \cdot [\mathbf{N}(\mathbf{r}(t)) - mg \mathbf{k}] + mg \mathbf{v}(t) \cdot \mathbf{k} = 0
\]
since \( \mathbf{v}(t) \cdot \mathbf{N}(\mathbf{r}(t)) = 0 \). So \( E(t) \) is a constant, independent of \( t \).
(b) By part (a),

\[ E(t) = E(0) \implies \frac{1}{2}m|\mathbf{v}(t)|^2 + mgb\theta(t) = mg(2\pi b) \implies |\mathbf{v}(t)|^2 = 2gb(2\pi - \theta(t)) \]

(c) Now

\[ \mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta} \frac{d\theta}{dt} = (-a \sin \theta, a \cos \theta, b) \frac{d\theta}{dt} \implies |\mathbf{v}|^2 = (a^2 + b^2)\left(\frac{d\theta}{dt}\right)^2 \]

\[ \implies \frac{d\theta}{dt} = -\left[\frac{|\mathbf{v}|^2}{a^2 + b^2}\right]^{1/2} = -\left[\frac{2gb(2\pi - \theta)}{a^2 + b^2}\right]^{1/2} \]

We have chosen the negative sign because \( \theta \) must decrease from \( 2\pi \) to 0. The time required to do so is

\[ \int dt = \int_{2\pi}^{0} \frac{d\theta}{dt} \quad \text{d}t = -\left[\frac{a^2 + b^2}{2gb}\right]^{1/2} \int_{2\pi}^{0} \frac{1}{(2\pi - \theta)^{1/2}} \quad d\theta = \left[\frac{a^2 + b^2}{2gb}\right]^{1/2} \int_{0}^{2\pi} \frac{1}{(2\pi - \theta)^{1/2}} \quad d\theta \]

\[ = \left[\frac{a^2 + b^2}{2gb}\right]^{1/2} \left[ -2(2\pi - \theta)^{1/2} \right]_{0}^{2\pi} = 2\left[\frac{a^2 + b^2}{gb}\right]^{1/2} \]