Math 227 Problem Set II Solutions

1. Find the curvature of the plane curve $y = e^x$.

   Solution. This curve has equation $y = f(x)$ with $f(x) = e^x$. So

$$\kappa(x) = \frac{f''(x)}{[1 + f'(x)^2]^{3/2}} = \frac{e^x}{[1 + e^{2x}]^{3/2}}$$

2. Find the minimum and maximum values for the curvature of the ellipse $x(t) = a\cos t$, $y(t) = b\sin t$. Here $a > b > 0$.

   Solution. For the given ellipse

$$\begin{aligned}
\mathbf{r}(t) &= a\cos t\,\hat{\boldsymbol{\imath}} + b\sin t\,\hat{\boldsymbol{\jmath}} \\
\mathbf{v}(t) &= -a\sin t\,\hat{\boldsymbol{\imath}} + b\cos t\,\hat{\boldsymbol{\jmath}} \qquad\qquad |\mathbf{v}(t)| = \sqrt{a^2\sin^2 t + b^2\cos^2 t} \\
\mathbf{a}(t) &= -a\cos t\,\hat{\boldsymbol{\imath}} - b\sin t\,\hat{\boldsymbol{\jmath}} \qquad\qquad \mathbf{v}(t)\times\mathbf{a}(t) = ab\,\hat{\mathbf{k}} \\
\kappa(t) &= \frac{|\mathbf{v}(t)\times\mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{ab}{[a^2\sin^2 t + b^2\cos^2 t]^{3/2}}
\end{aligned}$$

   Hence the maximum (minimum) curvature is achieved when the denominator is a minimum (maximum) which is the case when $\sin t = 0$ ($\cos t = 0$). So $\kappa_{\max} = \frac{a}{b^2}$ and $\kappa_{\min} = \frac{b}{a^2}$.

3. Let $r = f(\theta)$ be the equation of a plane curve in polar coordinates. Find the curvature of this curve at a general point θ.

   Solution. Think of θ as a time parameter. Then the given curve has

$$\begin{aligned}
\mathbf{r}(\theta) &= f(\theta)\big[\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}\big] \\
\mathbf{v}(\theta) &= f'(\theta)\big[\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}\big] + f(\theta)\big[-\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}\big] \\
|\mathbf{v}(\theta)| &= \sqrt{f'(\theta)^2 + f(\theta)^2} \\
\mathbf{a}(\theta) &= \big\{f''(\theta) - f(\theta)\big\}\big[\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}\big] + 2f'(\theta)\big[-\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}\big] \\
\mathbf{v}(\theta)\times\mathbf{a}(\theta) &= \big\{2f'(\theta)^2 - f(\theta)[f''(\theta) - f(\theta)]\big\}\,\hat{\mathbf{k}} \\
\kappa(\theta) &= \frac{|\mathbf{v}(\theta)\times\mathbf{a}(\theta)|}{|\mathbf{v}(\theta)|^3} = \frac{\big|f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta)\big|}{[f(\theta)^2 + f'(\theta)^2]^{3/2}}
\end{aligned}$$

   The computation of the cross product $\mathbf{v}(\theta)\times\mathbf{a}(\theta)$ (and also of $|\mathbf{v}(\theta)|$) has been facilitated by the observation that $\hat{\mathbf{e}}_r(\theta) = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}$ and $\hat{\mathbf{e}}_\theta(\theta) = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$ are mutually perpendicular unit vectors obeying $\hat{\mathbf{e}}_r(\theta)\times\hat{\mathbf{e}}_\theta(\theta) = \hat{\mathbf{k}}$ and $\hat{\mathbf{e}}_r(\theta)\times\hat{\mathbf{e}}_r(\theta) = \hat{\mathbf{e}}_\theta(\theta)\times\hat{\mathbf{e}}_\theta(\theta) = 0$.

4. Find the curvature of the cardiod $r = a(1 - \cos\theta)$.

   Solution. By the previous problem with

$$f(\theta) = a(1 - \cos\theta) \qquad f'(\theta) = a\sin\theta \qquad f''(\theta) = a\cos\theta$$

   we have

$$\begin{aligned}
\kappa(\theta) &= \frac{\big|f(\theta)^2 + 2f'(\theta)^2 - f(\theta)f''(\theta)\big|}{[f(\theta)^2 + f'(\theta)^2]^{3/2}} \\
&= \frac{\big|a^2 - 2a^2\cos\theta + a^2\cos^2\theta + 2a^2\sin^2\theta - a^2\cos\theta + a^2\cos^2\theta\big|}{[a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta]^{3/2}} \\
&= \frac{3a^2 - 3a^2\cos\theta}{[2a^2 - 2a^2\cos\theta]^{3/2}} = \frac{3}{2^{3/2}a\sqrt{1 - \cos\theta}} = \frac{3}{2\sqrt{2ar(\theta)}}
\end{aligned}$$

5. Find the unit tangent, unit normal and binormal vectors and the curvature and torsion of the curve

$$\mathbf{r}(t) = t\,\hat{\boldsymbol{\imath}} + \tfrac{t^2}{2}\,\hat{\boldsymbol{\jmath}} + \tfrac{t^3}{3}\,\hat{\mathbf{k}}$$

Solution. For the specified curve

$$\begin{aligned}
\mathbf{r}(t) &= t\,\hat{\boldsymbol{\imath}} + \tfrac{t^2}{2}\,\hat{\boldsymbol{\jmath}} + \tfrac{t^3}{3}\,\hat{\mathbf{k}} \\
\mathbf{v}(t) &= \hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}} + t^2\,\hat{\mathbf{k}} \\
\mathbf{a}(t) &= \hat{\boldsymbol{\jmath}} + 2t\,\hat{\mathbf{k}} \\
\mathbf{v}(\theta) \times \mathbf{a}(\theta) &= t^2\,\hat{\boldsymbol{\imath}} - 2t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}} \\
\mathbf{a}'(t) &= 2\,\hat{\mathbf{k}}
\end{aligned}$$

From this, we read off

$$\begin{aligned}
\hat{\mathbf{T}}(t) &= \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\hat{\boldsymbol{\imath}} + t\,\hat{\boldsymbol{\jmath}} + t^2\,\hat{\mathbf{k}}}{\sqrt{1+t^2+t^4}} \\
\kappa(t) &= \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{\sqrt{1+4t^2+t^4}}{[1+t^2+t^4]^{3/2}} \\
\hat{\mathbf{B}}(t) &= \frac{\mathbf{v}(t) \times \mathbf{a}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|} = \frac{t^2\,\hat{\boldsymbol{\imath}} - 2t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}}{\sqrt{1+4t^2+t^4}} \\
\hat{\mathbf{N}}(t) &= \hat{\mathbf{B}}(t) \times \hat{\mathbf{T}}(t) = \frac{-(t+2t^3)\,\hat{\boldsymbol{\imath}} + (1-t^4)\,\hat{\boldsymbol{\jmath}} + (2t+t^3)\hat{\mathbf{k}}}{\sqrt{1+t^2+t^4}\sqrt{1+4t^2+t^4}} \\
\tau(t) &= \frac{\big(\mathbf{v}(t) \times \mathbf{a}(t)\big) \cdot \mathbf{a}'(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|^2} = \frac{2}{1+4t^2+t^4}
\end{aligned}$$

6. (a) Show that if a curve has curvature $\kappa(s) = 0$ for all s, then the curve is a straight line.
(b) Show that if a curve has torsion $\tau(s) = 0$ for all s, then the curve lies in a plane.
(c) Show that if a curve has curvature $\kappa(s) = \kappa_0$, a strictly positive constant, and torsion $\tau(s) = 0$ for all s, then the curve is a circle.

Solution. (a) If $\kappa(s) \equiv 0$, then $\frac{d\hat{\mathbf{T}}}{ds} = \kappa(s)\hat{\mathbf{N}}(s) \equiv 0$ so that $\hat{\mathbf{T}}$ is a constant. As a result $\frac{d\mathbf{r}}{ds}(s) = \hat{\mathbf{T}}$ and $\mathbf{r}(s) = s\hat{\mathbf{T}} + \mathbf{r}(0)$ so that the curve is the straight line with direction vector $\hat{\mathbf{T}}$ that passes through $\mathbf{r}(0)$.
(b) If $\tau(s) \equiv 0$, then $\frac{d\hat{\mathbf{B}}}{ds} = -\tau(s)\hat{\mathbf{N}}(s) \equiv 0$ so that $\hat{\mathbf{B}}$ is a constant. As $\hat{\mathbf{T}}(s) \perp \hat{\mathbf{B}}$,

$$\tfrac{d}{ds}(\mathbf{r}(s) - \mathbf{r}(0)) \cdot \hat{\mathbf{B}} = \hat{\mathbf{T}}(s) \cdot \hat{\mathbf{B}} = 0$$

and $(\mathbf{r}(s) - \mathbf{r}(0)) \cdot \hat{\mathbf{B}}$ must be a constant. The constant must be zero (set $s = 0$), so $(\mathbf{r}(s) - \mathbf{r}(0)) \cdot \hat{\mathbf{B}} = 0$ and $\mathbf{r}(s)$ always lies in the plane through $\mathbf{r}(0)$ with normal vector $\hat{\mathbf{B}}$.
(c) Parametrize the curve by arc length. Define the "centre of curvature" at s by

$$\mathbf{r}_c(s) = \mathbf{r}(s) + \tfrac{1}{\kappa(s)}\hat{\mathbf{N}}(s)$$

Since $\kappa(s) = \kappa_0$ is a constant and $\tau(s) \equiv 0$,

$$\tfrac{d}{ds}\mathbf{r}_c(s) = \hat{\mathbf{T}}(s) + \tfrac{1}{\kappa_0}\big[\tau(s)\hat{\mathbf{B}} - \kappa(s)\hat{\mathbf{T}}\big] = \hat{\mathbf{T}}(s) + \tfrac{1}{\kappa_0}\big[0\hat{\mathbf{B}} - \kappa_0\hat{\mathbf{T}}\big] = 0$$

Thus $\mathbf{r}_c(s) = \mathbf{r}_c$ is a consant and $\big|\mathbf{r}(s) - \mathbf{r}_c\big| = \frac{1}{\kappa_0}$ lies on the sphere of radius $\frac{1}{\kappa_0}$ centred on $\mathbf{r}_c$. Since $\tau(s) \equiv 0$, the curve also lies on a plane, so it is a circle.

7. Solve the initial value problem

$$\tfrac{d\mathbf{r}}{dt} = \hat{\mathbf{k}} \times \mathbf{r} \qquad\qquad \mathbf{r}(0) = \hat{\boldsymbol{\imath}} + \hat{\mathbf{k}}$$

Describe the curve $\mathbf{r} = \mathbf{r}(t)$.

Solution. The velocity vector $\frac{d\mathbf{r}}{dt} = \hat{\mathbf{k}} \times \mathbf{r}$ is always perpendicular to $\hat{\mathbf{k}}$. Consequently, the curve lies in a plane perpendicular to $\hat{\mathbf{k}}$. Furthermore, the velocity vector is always perpendicular to $\mathbf{r}$. So the curve lies on a sphere. (See problem #7 of Problem Set I.) The intersection of a sphere with a plane is a circle, so the path is a circle. We are told that $\mathbf{r}(0) = \hat{\boldsymbol{\imath}} + \hat{\mathbf{k}}$. Guess $\mathbf{r}(t) = \cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$. As

$$\begin{aligned}
\cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\Big|_{t=0} &= \hat{\boldsymbol{\imath}} + \hat{\mathbf{k}} \\
\tfrac{d}{dt}\big[\cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big] &= -\sin t\,\hat{\boldsymbol{\imath}} + \cos t\,\hat{\boldsymbol{\jmath}} \\
\hat{\mathbf{k}} \times \big[\cos t\,\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\big] &= \cos t\,\hat{\boldsymbol{\jmath}} - \sin t\,\hat{\boldsymbol{\imath}}
\end{aligned}$$

the guess satisfies both the required differential equation and the required initial condition.

8. Find the curvature κ as a function of arclength s (measured from $(0,0)$) for the curve

$$x(\theta) = \int_0^\theta \cos\left(\tfrac{1}{2}\pi t^2\right) dt \qquad y(\theta) = \int_0^\theta \sin\left(\tfrac{1}{2}\pi t^2\right) dt$$

Solution. The velocity vector is

$$\mathbf{v}(\theta) = x'(\theta)\,\hat{\boldsymbol{\imath}} + y'(\theta)\,\hat{\boldsymbol{\jmath}} = \cos\left(\tfrac{1}{2}\pi\theta^2\right)\hat{\boldsymbol{\imath}} + \sin\left(\tfrac{1}{2}\pi\theta^2\right)\hat{\boldsymbol{\jmath}}$$

Consequently the speed

$$\tfrac{ds}{d\theta}(\theta) = |\mathbf{v}(\theta)| = 1 \quad\Longrightarrow\quad s(\theta) = \theta + s(0)$$

Since $s(\theta)$ is zero when $\theta = 0$, we have $s(\theta) = \theta$ and hence $\hat{\mathbf{T}}(s) = \mathbf{v}(s) = \cos\left(\tfrac{1}{2}\pi s^2\right)\hat{\boldsymbol{\imath}} + \sin\left(\tfrac{1}{2}\pi s^2\right)\hat{\boldsymbol{\jmath}}$ so that

$$\kappa(s) = \left|\tfrac{d\hat{\mathbf{T}}}{ds}(s)\right| = \left| -\pi s \sin\left(\tfrac{1}{2}\pi s^2\right)\hat{\boldsymbol{\imath}} + \pi s \cos\left(\tfrac{1}{2}\pi s^2\right)\hat{\boldsymbol{\jmath}}\right| = \pi s$$

9. A frictionless roller–coaster track has the form of one turn of the circular helix with parametrization $(a\cos\theta, a\sin\theta, b\theta)$. A car leaves the point where $\theta = 2\pi$ with zero velocity and moves under gravity to the point where $\theta = 0$. By Newton's law of motion, the position $\mathbf{r}(t)$ of the car at time t obeys

$$m\mathbf{r}''(t) = \mathbf{N}\big(\mathbf{r}(t)\big) - mg\hat{\mathbf{k}}$$

Here m is the mass of the car, g is a constant, $-mg\hat{\mathbf{k}}$ is the force due to gravity and $\mathbf{N}\big(\mathbf{r}(t)\big)$ is the force that the roller–coaster track applies to the car to keep the car on the track. Since the track is frictionless, $\mathbf{N}\big(\mathbf{r}(t)\big)$ is always perpendicular to $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}(t)$.

(a) Prove that $E(t) = \frac{1}{2}m|\mathbf{v}(t)|^2 + mg\mathbf{r}(t)\cdot\hat{\mathbf{k}}$ is a constant, independent of t. (This is called "conservation of energy".)

(b) Prove that the speed $|\mathbf{v}|$ at the point θ obeys $|\mathbf{v}|^2 = 2gb(2\pi - \theta)$.

(c) Find the time it takes to reach $\theta = 0$.

Solution. (a) By Newton's law of motion

$$\begin{aligned}
E'(t) &= \tfrac{d}{dt}\left[\tfrac{1}{2}m|\mathbf{v}(t)|^2 + mg\mathbf{r}(t)\cdot\hat{\mathbf{k}}\right] = m\mathbf{v}(t)\cdot\mathbf{v}'(t) + mg\mathbf{v}(t)\cdot\hat{\mathbf{k}} = \mathbf{v}(t)\cdot\left[\mathbf{N}\big(\mathbf{r}(t)\big) - mg\hat{\mathbf{k}}\right] + mg\mathbf{v}(t)\cdot\hat{\mathbf{k}} \\
&= 0
\end{aligned}$$

since $\mathbf{v}(t)\cdot\mathbf{N}\big(\mathbf{r}(t)\big) = 0$. So $E(t)$ is a constant, independent of t.

(b) By part (a),

$$E(t) = E(0) \quad \Longrightarrow \quad \tfrac{1}{2}m|\mathbf{v}(t)|^2 + mgb\theta(t) = mg(2\pi b) \quad \Longrightarrow \quad |\mathbf{v}(t)|^2 = 2gb\big(2\pi - \theta(t)\big)$$

(c) Now

$$\mathbf{v} = \tfrac{d\mathbf{r}}{dt} = \tfrac{d\mathbf{r}}{d\theta}\tfrac{d\theta}{dt} = \big(-a\sin\theta, a\cos\theta, b\big)\tfrac{d\theta}{dt} \quad \Longrightarrow \quad |\mathbf{v}|^2 = \big[a^2+b^2\big]\big(\tfrac{d\theta}{dt}\big)^2$$
$$\Longrightarrow \quad \tfrac{d\theta}{dt} = -\big[\tfrac{|\mathbf{v}|^2}{a^2+b^2}\big]^{1/2} = -\big[\tfrac{2gb(2\pi-\theta)}{a^2+b^2}\big]^{1/2}$$

We have chosen the negative sign because θ must decrease from 2π to 0. The time required to do so is

$$\int dt = \int_{2\pi}^{0} \tfrac{dt}{d\theta}d\theta = -\big[\tfrac{a^2+b^2}{2gb}\big]^{1/2}\int_{2\pi}^{0}\tfrac{1}{(2\pi-\theta)^{1/2}}d\theta = \big[\tfrac{a^2+b^2}{2gb}\big]^{1/2}\int_{0}^{2\pi}\tfrac{1}{(2\pi-\theta)^{1/2}}d\theta$$
$$= \big[\tfrac{a^2+b^2}{2gb}\big]^{1/2}\big[-2(2\pi-\theta)^{1/2}\big]_0^{2\pi} = 2\big[\tfrac{a^2+b^2}{gb}\pi\big]^{1/2}$$