Math 227 Problem Set I Solutions

1. Find the velocity, speed and acceleration at time $t$ of the particle whose position is $\vec{r}(t)$. Describe the path of the particle.
   a) $\vec{r}(t) = a \cos t \, \hat{i} + a \sin t \, \hat{j} + c t \, \hat{k}$
   b) $\vec{r}(t) = a \cos t \sin t \, \hat{i} + a \sin^2 t \, \hat{j} + a \cos t \, \hat{k}$

   **Solution.** a)
   $$\vec{r}(t) = a \cos t \, \hat{i} + a \sin t \, \hat{j} + c t \, \hat{k}$$
   $$\vec{v}(t) = -a \sin t \, \hat{i} + a \cos t \, \hat{j} + c \, \hat{k}$$
   $$\vec{a}(t) = -a \cos t \, \hat{i} - a \sin t \, \hat{j}$$

   The path is a helix with radius $a$ and with each turn having height $2\pi c$.

   b)
   $$\vec{r}(t) = a \cos t \sin t \, \hat{i} + a \sin^2 t \, \hat{j} + a \cos t \, \hat{k}$$
   $$\vec{v}(t) = a \cos 2t \, \hat{i} + a \sin 2t \, \hat{j} - a \sin t \, \hat{k}$$
   $$\vec{a}(t) = -2a \sin 2t \, \hat{i} + 2a \cos 2t \, \hat{j} - a \cos t \, \hat{k}$$

   The $(x, y)$ coordinates go around a circle of radius $\frac{a}{2}$ and centre $(0, \frac{a}{2})$ counterclockwise. At the same time the $z$ coordinate oscillates half as fast. As $t$ runs from 0 to $\pi$,
   - $z$ starts at $0$ when $(x, y)$ is at $(0, 0)$, then decreases to $0$ when $(x, y)$ is at $(0, a)$, and continues decreasing to $-a$ when $(x, y)$ is back at $(0, 0)$.

   Then as $t$ runs from $\pi$ to $2\pi$,
   - $z$ starts at $-a$ when $(x, y)$ is at $(0, 0)$, then increases to $0$ when $(x, y)$ is at $(0, a)$, and continues increasing to $a$ when $(x, y)$ is back at $(0, 0)$.

   Then the pattern repeats.

2. A projectile falling under the influence of gravity and slowed by air resistance proportional to its speed has position satisfying
   $$\frac{d^2 \vec{r}}{dt^2} = -g \, \hat{k} - \alpha \frac{d\vec{r}}{dt}$$

   where $\alpha$ is a positive constant. If $\vec{r} = \vec{r}_0$ and $\frac{d\vec{r}}{dt} = \vec{v}_0$ at time $t = 0$, find $\vec{r}(t)$. (Hint: consider $\vec{u}(t) = e^{\alpha t} \frac{d\vec{r}}{dt}(t)$.)

   **Solution.** Define $\vec{u}(t) = e^{\alpha t} \frac{d\vec{r}}{dt}(t)$. Then
   $$\frac{d\vec{u}}{dt}(t) = \alpha e^{\alpha t} \frac{d\vec{r}}{dt}(t) + e^{\alpha t} \frac{d^2\vec{r}}{dt^2}(t)$$
   $$= \alpha e^{\alpha t} \frac{d\vec{r}}{dt}(t) - g e^{\alpha t} \hat{k} - \alpha e^{\alpha t} \frac{d\vec{r}}{dt}(t)$$
   $$= -g e^{\alpha t} \hat{k}$$

   Integrating both sides of this equation from $t = 0$ to $t = T$ gives
   $$\vec{u}(T) - \vec{u}(0) = -g \frac{e^{\alpha T} - 1}{\alpha} \hat{k} \Rightarrow \vec{u}(T) = \vec{u}(0) - g \frac{e^{\alpha T} - 1}{\alpha} \hat{k} = \frac{d\vec{r}}{dt}(0) - g \frac{e^{\alpha T} - 1}{\alpha} \hat{k} = \vec{v}_0 - g \frac{e^{\alpha T} - 1}{\alpha} \hat{k}$$

   Subbing in $\vec{u}(T) = e^{\alpha t} \frac{d\vec{r}}{dt}(T)$ and multiplying through by $e^{-\alpha T}$
   $$\frac{d\vec{r}}{dt}(T) = e^{-\alpha T} \vec{v}_0 - g \frac{e^{-\alpha T} - 1}{\alpha} \hat{k}$$
Integrating both sides of this equation from $T = 0$ to $T = t$ gives
\[
\vec{r}(t) - \vec{r}(0) = \frac{e^{-at} - 1}{a} \vec{r}_0 - g \frac{t}{a} \hat{k} + g e^{-at} \vec{r}_0 \Rightarrow \vec{r}(t) = \vec{r}_0 - \frac{e^{-at} - 1}{a} \vec{r}_0 + g \frac{1 - e^{-at}}{a^2} \hat{k}
\]

3. Find the specified parametrization of the first quadrant part of the circle $x^2 + y^2 = a^2$.
   a) In terms of the $y$ coordinate.
   b) In terms of the angle between the tangent line and the positive $x$-axis.
   c) In terms of the arc length from $(0, a)$.

**Solution.**

a) \( (x(t), y(t)) = (\sqrt{a^2 - t^2}, t), \quad 0 \leq t \leq a \).  

b) Let $\theta$ be the angle between the radius vector \((a \cos \theta, a \sin \theta)\) and the positive $x$-axis. The tangent line to the circle at \((a \cos \theta, a \sin \theta)\) is perpendicular to the radius vector and so makes angle $\phi = \frac{\pi}{2} + \theta$ with the positive $x$ axis. The desired parametrization is
\[
(x(\phi), y(\phi)) = (a \cos(\phi - \frac{\pi}{2}), a \sin(\phi - \frac{\pi}{2})) = (a \sin \phi, -a \cos \phi), \quad \frac{\pi}{2} \leq \phi \leq \pi
\]

c) Let $\theta$ be the angle between the radius vector \((a \cos \theta, a \sin \theta)\) and the positive $x$-axis. The arc from $(0, a)$ to $(a \cos \theta, a \sin \theta)$ subtends an angle $\frac{\pi}{2} - \theta$ and so has length $s = a \left( \frac{\pi}{2} - \theta \right)$. Thus $\theta = \frac{\pi}{2} - \frac{s}{a}$ and the desired parametrization is
\[
(x(\phi), y(\phi)) = (a \cos \left( \frac{\pi}{2} - \frac{s}{a} \right), a \sin \left( \frac{\pi}{2} - \frac{s}{a} \right)), \quad 0 \leq s \leq \frac{\pi}{2} a
\]

4. Find the length of the parametric curve
\[
x = a \cos t \sin t \quad y = a \sin^2 t \quad z = bt
\]
between $t = 0$ and $t = T > 0$.

**Solution.**
\[
x'(t) = a \left[ \cos^2 t - \sin^2 t \right] = a \cos 2t
\]
\[
y'(t) = 2a \sin t \cos t = a \sin 2t
\]
\[
z'(t) = b
\]
So
\[
\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{a^2 + b^2} \quad \Rightarrow \quad \text{length} = \sqrt{a^2 + b^2} T
\]

5. Reparametrize the curve
\[
\vec{r}(t) = a \cos^3 t \hat{i} + a \sin^3 t \hat{j} + b \cos 2t \hat{k}, \quad 0 \leq t \leq \frac{\pi}{2}
\]
with the same orientation, in terms of arc length measured from the point where $t = 0$.

**Solution.**
\[
\vec{v}(t) = \left( -3a \cos^2 t \sin t, 3a \sin^2 t \cos t, -2b \sin 2t \right)
\]
\[
\Rightarrow \quad \frac{ds}{dt} = ||\vec{v}(t)|| = \left[ 9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t + 4b^2 \sin^2 2t \right]^{1/2}
\]
\[
= \left[ 9a^2 \sin^2 t \cos^2 t + 4b^2 \sin^2 2t \right]^{1/2}
\]
\[
= \left[ \frac{a^2}{4} + 4b^2 \right]^{1/2} |\sin 2t| = \left[ \frac{a^2}{4} + 4b^2 \right]^{1/2} \sin 2t \text{ for } 0 \leq t \leq \frac{\pi}{4}
\]
Integrating (and recalling that \( s = 0 \) corresponds to \( t = 0 \))

\[
s(t) = \left[ \frac{9}{4} a^2 + 4b^2 \right]^{1/2} \frac{1 - \cos 2t}{2} = \left[ \frac{9}{4} a^2 + 4b^2 \right]^{1/2} \sin^2 t
\]

Setting \( K = \left[ \frac{9}{4} a^2 + 4b^2 \right]^{1/2} \), we have \( \sin t = \frac{K}{\sqrt{K}} \), \( \cos t = \sqrt{1 - \frac{K}{K}} \), \( \cos 2t = 1 - 2\sin^2 t = 1 - \frac{2K}{K} \) and hence

\[
\vec{r}(s) = a\left[ 1 - \frac{s}{K} \right]^{3/2} \hat{i} + a\left[ 1 - \frac{s}{K} \right]^{3/2} \hat{j} + b\left[ 1 - \frac{s}{K} \right] \hat{k}, \quad 0 \leq s \leq K,
\]

where \( K = \left[ \frac{9}{4} a^2 + 4b^2 \right]^{1/2} \)

6. The plane \( z = 2x + 3y \) intersects the cylinder \( x^2 + y^2 = 9 \) in an ellipse. Find a parametrization of the ellipse. Express the circumference of this ellipse as an integral. You need not evaluate the integral.

Solution. \( x = 3 \cos \theta, \ y = 3 \sin \theta, \ z = 6 \cos \theta + 9 \sin \theta, \ 0 \leq \theta \leq 2\pi \)

\[
\frac{ds}{d\theta} = \sqrt{x'(\theta)^2 + y'(\theta)^2 + z'(\theta)^2} = \sqrt{9 \sin^2 \theta + 9 \cos^2 \theta + 36 \sin^2 \theta + 81 \cos^2 \theta - 108 \sin \theta \cos \theta} = \sqrt{45 + 45 \cos^2 \theta - 108 \sin \theta \cos \theta}
\]

\[
\Rightarrow \quad s = \int_0^{2\pi} \sqrt{45 + 45 \cos^2 \theta - 108 \sin \theta \cos \theta} d\theta
\]

7. A wire of total length 1000 cm is formed into a flexible coil that is a circular helix. If there are 10 turns to each centimeter of height and the radius of the helix is 3 cm, how tall is the coil?

Solution. The parametrized equation of a helix is

\[
\vec{r}(\theta) = a \cos \theta \hat{i} + a \sin \theta \hat{j} + b \theta \hat{k}
\]

The radius of the helix is 3 cm, so \( a = 3 \) cm. After 10 turns (i.e. \( \theta = 20\pi \)) the height is 1 cm, so \( b = \frac{1}{20\pi} \) cm/rad. Thus \( \vec{r}(\theta) = 3 \cos \theta \hat{i} + 3 \sin \theta \hat{j} + \frac{1}{20\pi} \theta \hat{k} \) and \( \vec{r}'(\theta) = -3 \sin \theta \hat{i} + 3 \cos \theta \hat{j} + \frac{1}{20\pi} \hat{k} \) so that \( \frac{ds}{d\theta} = ||\vec{r}'(\theta)|| = \sqrt{9 + \frac{1}{400\pi^2}} \). If \( \theta \) varies from \( \theta = 0 \) to \( \theta = \theta_F \), then the wire has length \( \sqrt{9 + \frac{1}{400\pi^2}} \theta_F \).

This must be 1000 cm. So \( \theta_F = 1000 \left[ 9 + \frac{1}{400\pi^2} \right]^{-1/2} \). This corresponds to a height

\[
b \theta_F = \frac{1}{20\pi} 1000 \left[ 9 + \frac{1}{400\pi^2} \right]^{-1/2} \approx 5.3 \text{ cm}
\]

8. Show that, if the position and velocity vectors of a moving particle are always perpendicular, then the path of the particle lies on a sphere.

Solution.

\[
\frac{d}{dt} ||\vec{r}(t)||^2 = \frac{d}{dt} [\vec{r}(t) \cdot \vec{r}(t)] = 2\vec{r}(t) \cdot \vec{r}'(t) = 0
\]

since we are told that \( \vec{r}(t) \perp \vec{r}'(t) \) for all \( t \). Consequently, \( ||\vec{r}(t)||^2 \) is a constant, say \( A \), independent of time and \( \vec{r}(t) \) always lies on the sphere of radius \( \sqrt{A} \) centred on the origin.