Math 227 Problem Set I Solutions

1. Find the velocity, speed and acceleration at time t of the particle whose position is $\vec{r}(t)$. Describe the path of the particle.
   a) $\vec{r}(t) = a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + ct\,\hat{k}$
   b) $\vec{r}(t) = a\cos t\sin t\,\hat{\imath} + a\sin^2 t\,\hat{\jmath} + a\cos t\,\hat{k}$

   Solution. a)

$$\begin{aligned}\vec{r}(t) &= a\cos t\,\hat{\imath} + a\sin t\,\hat{\jmath} + ct\,\hat{k}\\ \vec{v}(t) &= -a\sin t\,\hat{\imath} + a\cos t\,\hat{\jmath} + c\,\hat{k}\\ \vec{a}(t) &= -a\cos t\,\hat{\imath} - a\sin t\,\hat{\jmath}\end{aligned}$$

   The path is a helix with radius a and with each turn having height $2\pi c$.
   b)

$$\begin{aligned}\vec{r}(t) &= a\cos t\sin t\,\hat{\imath} + a\sin^2 t\,\hat{\jmath} + a\cos t\,\hat{k}\\ &= \tfrac{a}{2}\sin 2t\,\hat{\imath} + a\tfrac{1-\cos 2t}{2}\,\hat{\jmath} + a\cos t\,\hat{k}\\ \vec{v}(t) &= a\cos 2t\,\hat{\imath} + a\sin 2t\,\hat{\jmath} - a\sin t\,\hat{k}\\ \vec{a}(t) &= -2a\sin 2t\,\hat{\imath} + 2a\cos 2t\,\hat{\jmath} - a\cos t\,\hat{k}\end{aligned}$$

   The (x, y) coordinates go around a circle of radius $\frac{a}{2}$ and centre $\left(0, \frac{a}{2}\right)$ counterclockwise. At the same time the z coordinate oscillates half as fast. As t runs from 0 to π,
   - the (x, y) coordinates go around the circle once, starting at $(0, 0)$, while
   - z starts at a when (x, y) is at $(0, 0)$, then decreases to 0 when (x, y) is at $(0, a)$, and continues decreasing to $-a$ when (x, y) is back at $(0, 0)$.

   Then as t runs from π to 2π,
   - the (x, y) coordinates go around the circle a second time, starting at $(0, 0)$, while
   - z starts at $-a$ when (x, y) is at $(0, 0)$, then increases to 0 when (x, y) is at $(0, a)$, and continues increasing to a when (x, y) is back at $(0, 0)$.

   Then the pattern repeats.

2. A projectile falling under the influence of gravity and slowed by air resistance proportional to its speed has position satisfying

$$\frac{d^2\vec{r}}{dt^2} = -g\hat{k} - \alpha\frac{d\vec{r}}{dt}$$

   where α is a positive constant. If $\vec{r} = \vec{r}_0$ and $\frac{d\vec{r}}{dt} = \vec{v}_0$ at time $t = 0$, find $\vec{r}(t)$. (Hint: consider $\vec{u}(t) = e^{\alpha t}\frac{d\vec{r}}{dt}(t)$.)

   Solution. Define $\vec{u}(t) = e^{\alpha t}\frac{d\vec{r}}{dt}(t)$. Then

$$\begin{aligned}\frac{d\vec{u}}{dt}(t) &= \alpha e^{\alpha t}\frac{d\vec{r}}{dt}(t) + e^{\alpha t}\frac{d^2\vec{r}}{dt^2}(t)\\ &= \alpha e^{\alpha t}\frac{d\vec{r}}{dt}(t) - ge^{\alpha t}\hat{k} - \alpha e^{\alpha t}\frac{d\vec{r}}{dt}(t)\\ &= -ge^{\alpha t}\hat{k}\end{aligned}$$

   Integrating both sides of this equation from $t = 0$ to $t = T$ gives

$$\vec{u}(T) - \vec{u}(0) = -g\frac{e^{\alpha T}-1}{\alpha}\hat{k} \;\Rightarrow\; \vec{u}(T) = \vec{u}(0) - g\frac{e^{\alpha T}-1}{\alpha}\hat{k} = \frac{d\vec{r}}{dt}(0) - g\frac{e^{\alpha T}-1}{\alpha}\hat{k} = \vec{v}_0 - g\frac{e^{\alpha T}-1}{\alpha}\hat{k}$$

   Subbing in $\vec{u}(T) = e^{\alpha t}\frac{d\vec{r}}{dt}(T)$ and multiplying through by $e^{-\alpha T}$

$$\frac{d\vec{r}}{dt}(T) = e^{-\alpha T}\vec{v}_0 - g\frac{1-e^{-\alpha T}}{\alpha}\hat{k}$$

Integrating both sides of this equation from $T = 0$ to $T = t$ gives

$$\vec{r}(t) - \vec{r}(0) = \tfrac{e^{-\alpha t}-1}{-\alpha}\vec{v}_0 - g\tfrac{t}{\alpha}\hat{k} + g\tfrac{e^{-\alpha t}-1}{-\alpha^2}\hat{k}$$
$$\Longrightarrow \quad \boxed{\vec{r}(t) = \vec{r}_0 - \tfrac{e^{-\alpha t}-1}{\alpha}\vec{v}_0 + g\tfrac{1-\alpha t-e^{-\alpha t}}{\alpha^2}\hat{k}}$$

3. Find the specified parametrization of the first quadrant part of the circle $x^2 + y^2 = a^2$.
   a) In terms of the y coordinate.
   b) In terms of the angle between the tangent line and the positive x–axis.
   c) In terms of the arc length from $(0, a)$.

   Solution. a) $\boxed{\big(x(t), y(t)\big) = \big(\sqrt{a^2 - t^2}, t\big),\ 0 \le t \le a}$.

   b) Let θ be the angle between the radius vector $(a\cos\theta, a\sin\theta)$ and the positive x–axis. The tangent line to the circle at $(a\cos\theta, a\sin\theta)$ is perpendicular to the radius vector and so makes angle $\phi = \frac{\pi}{2} + \theta$ with the positive x axis. The desired parametrization is

$$\boxed{\big(x(\phi), y(\phi)\big) = \big(a\cos(\phi - \tfrac{\pi}{2}), a\sin(\phi - \tfrac{\pi}{2})\big) = \big(a\sin\phi, -a\cos\phi\big),\ \tfrac{\pi}{2} \le \phi \le \pi}$$

   c) Let θ be the angle between the radius vector $(a\cos\theta, a\sin\theta)$ and the positive x–axis. The arc from $(0, a)$ to $(a\cos\theta, a\sin\theta)$ subtends an angle $\frac{\pi}{2} - \theta$ and so has length $s = a\big(\frac{\pi}{2} - \theta\big)$. Thus $\theta = \frac{\pi}{2} - \frac{s}{a}$ and the desired parametrization is

$$\boxed{\big(x(\phi), y(\phi)\big) = \big(a\cos(\tfrac{\pi}{2} - \tfrac{s}{a}), a\sin(\tfrac{\pi}{2} - \tfrac{s}{a})\big),\ 0 \le s \le \tfrac{\pi}{2}a}$$

4. Find the length of the parametric curve

$$x = a\cos t\sin t \qquad y = a\sin^2 t \qquad z = bt$$

   between $t = 0$ and $t = T > 0$.

   Solution.

$$\begin{aligned} x'(t) &= a\big[\cos^2 t - \sin^2 t\big] = a\cos 2t \\ y'(t) &= 2a\sin t\cos t = a\sin 2t \\ z'(t) &= b \end{aligned}$$

   So

$$\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{a^2 + b^2} \quad \Longrightarrow \quad \boxed{\text{length}= \sqrt{a^2 + b^2}\,T}$$

5. Reparametrize the curve

$$\vec{r}(t) = a\cos^3 t\,\hat{\imath} + a\sin^3 t\,\hat{\jmath} + b\cos 2t\,\hat{k},\ 0 \le t \le \tfrac{\pi}{2}$$

   with the same orientation, in terms of arc length measured from the point where $t = 0$.

   Solution.

$$\begin{aligned} \vec{v}(t) &= \big(-3a\cos^2 t\sin t, 3a\sin^2 t\cos t, -2b\sin 2t\big) \\ \Longrightarrow \quad \tfrac{ds}{dt} = \|\vec{v}(t)\| &= \big[9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t + 4b^2\sin^2 2t\big]^{1/2} \\ &= \big[9a^2\sin^2 t\cos^2 t + 4b^2\sin^2 2t\big]^{1/2} \\ &= \big[\tfrac{9}{4}a^2 + 4b^2\big]^{1/2}|\sin 2t| = \big[\tfrac{9}{4}a^2 + 4b^2\big]^{1/2}\sin 2t \text{ for } 0 \le t \le \tfrac{\pi}{2} \end{aligned}$$

Integrating (and recalling that $s = 0$ corresponds to $t = 0$)

$$s(t) = \left[\tfrac{9}{4}a^2 + 4b^2\right]^{1/2} \tfrac{1-\cos 2t}{2} = \left[\tfrac{9}{4}a^2 + 4b^2\right]^{1/2} \sin^2 t$$

Setting $K = \left[\frac{9}{4}a^2 + 4b^2\right]^{1/2}$, we have $\sin t = \sqrt{\frac{s}{K}}$, $\cos t = \sqrt{1 - \frac{s}{K}}$, $\cos 2t = 1 - 2\sin^2 t = 1 - \frac{2s}{K}$ and hence

$$\boxed{\vec{r}(s) = a\left[1 - \tfrac{s}{K}\right]^{3/2} \hat{\imath} + a\left[\tfrac{s}{K}\right]^{3/2} \hat{\jmath} + b\left[1 - \tfrac{2s}{K}\right] \hat{k},\ 0 \le s \le K,\ \text{where } K = \left[\tfrac{9}{4}a^2 + 4b^2\right]^{1/2}}$$

6. The plane $z = 2x + 3y$ intersects the cylinder $x^2 + y^2 = 9$ in an ellipse. Find a parametrization of the ellipse. Express the circumference of this ellipse as an integral. You need not evaluate the integral.

Solution.

$$\boxed{x = 3\cos\theta,\ y = 3\sin\theta,\ z = 6\cos\theta + 9\sin\theta,\ 0 \le \theta \le 2\pi}$$

$$\begin{aligned}
\tfrac{ds}{d\theta} &= \sqrt{x'(\theta)^2 + y'(\theta)^2 + z'(\theta)^2} \\
&= \sqrt{9\sin^2\theta + 9\cos^2\theta + 36\sin^2\theta + 81\cos^2\theta - 108\sin\theta\cos\theta} \\
&= \sqrt{45 + 45\cos^2\theta - 108\sin\theta\cos\theta} \\
&\implies \boxed{s = \textstyle\int_0^{2\pi} \sqrt{45 + 45\cos^2\theta - 108\sin\theta\cos\theta}\, d\theta}
\end{aligned}$$

7. A wire of total length 1000 cm is formed into a flexible coil that is a circular helix. If there are 10 turns to each centimeter of height and the radius of the helix is 3 cm, how tall is the coil?

Solution. The parametrized equation of a helix is

$$\vec{r}(\theta) = a\cos\theta\,\hat{\imath} + a\sin\theta\,\hat{\jmath} + b\theta\hat{k}$$

The radius of the helix is 3 cm, so $a = 3$ cm. After 10 turns (i.e. $\theta = 20\pi$) the height is 1 cm, so $b = \frac{1}{20\pi}$ cm/rad. Thus $\vec{r}(\theta) = 3\cos\theta\,\hat{\imath} + 3\sin\theta\,\hat{\jmath} + \frac{1}{20\pi}\theta\hat{k}$ and $\vec{r}'(\theta) = -3\sin\theta\,\hat{\imath} + 3\cos\theta\,\hat{\jmath} + \frac{1}{20\pi}\hat{k}$ so that $\frac{ds}{d\theta} = \|\vec{r}'(\theta)\| = \sqrt{9 + \frac{1}{400\pi^2}}$. If θ varies from $\theta = 0$ to $\theta = \theta_F$, then the wire has length $\sqrt{9 + \frac{1}{400\pi^2}}\theta_F$. This must be 1000 cm. So $\theta_F = 1000\left[9 + \frac{1}{400\pi^2}\right]^{-1/2}$. This corresponds to a height

$$b\theta_F = \tfrac{1}{20\pi}1000\left[9 + \tfrac{1}{400\pi^2}\right]^{-1/2} \approx \boxed{5.3 \text{ cm}}$$

8. Show that, if the position and velocity vectors of a moving particle are always perpendicular, then the path of the particle lies on a sphere.

Solution.

$$\tfrac{d}{dt}\|\vec{r}(t)\|^2 = \tfrac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = 2\vec{r}(t)\cdot\vec{r}'(t) = 0$$

since we are told that $\vec{r}(t) \perp \vec{r}'(t)$ for all t. Consequently, $\|\vec{r}(t)\|^2$ is a constant, say A, independent of time and $\vec{r}(t)$ always lies on the sphere of radius $\sqrt{A}$ centred on the origin.